In this chapter, you will learn to:

1. Write transition matrices for Markov Chain problems.

2. Explore some ways in which Markov Chains models are used in business, finance, public health and other fields of application

3. Find the long term trend for a Regular Markov Chain.

4. Solve and interpret Absorbing Markov Chains.

# 10.1 Introduction to Markov Chains

In this chapter, you will learn to:

1. Write transition matrices for Markov Chain problems.

2. Use the transition matrix and the initial state vector to find the state vector that gives the distribution after a specified number of transitions.

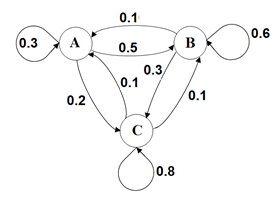
We will now study stochastic processes, experiments in which the outcomes of events depend on the previous outcomes; stochastic processes involve random outcomes that can be described by probabilities. Such a process or experiment is called a **Markov Chain** or **Markov process**. The process was first studied by a Russian mathematician named Andrei A. Markov in the early 1900s.

About 600 cities worldwide have bike share programs. Typically a person pays a fee to join a the program and can borrow a bicycle from any bike share station and then can return it to the same or another system. Each day, the distribution of bikes at the stations changes, as the bikes get returned to different stations from where they are borrowed.

For simplicity, let’s consider a very simple bike share program with only 3 stations: A, B, C. Suppose that all bicycles must be returned to the station at the end of the day, so that each day there is a time, let’s say midnight, that all bikes are at some station, and we can examine all the stations at this time of day, every day. We want to model the movement of bikes from midnight of a given day to midnight of the next day. We find that over a 1 day period,

* of the bikes borrowed from station A, 30% are returned to station A, 50% end up at station B, and 20% end up at station C.
* of the bikes borrowed from station B, 10% end up at station A, 60% have been returned to station B, and 30% end up at station C
* of the bikes borrowed from station C, 10% end up at station A, 10% end up at station B, and 80% are returned to station C.

We can draw an arrow diagram to show this. The arrows indicate the station where the bicycle was started, called its initial state, and the stations at which it might be located one day later, called the terminal states. The numbers on the arrows show the probability for being in each of the indicated terminal states.



Because our bike share example is simple and has only 3 stations, the arrow diagram, also called a directed graph, helps us visualize the information. But if we had an example with 10, or 20, or more bike share stations, the diagram would become so complicated that it would be difficult to understand the information in the diagram.

We can use a **transition matrix** to organize the information,

Each row in the matrix represents an initial state. Each column represents a terminal state.   
We will assign the rows in order to stations A, B, C, and the columns in the same order to stations A, B, C. Therefore the matrix must be a square matrix, with the same number of rows as columns. The entry in row 2 column 3, for example, would show the probability that a bike that is initially at station B will be at station C one day later: that entry is 0.30, which is the probability in the diagram for the arrow that points from B to C. We use this the letter T for transition matrix.

T = 

Looking at the first row that represents bikes initially at station A, we see that 30% of the bikes borrowed from station A are returned to station A, 50% end up at station B, and 20% end up at station C, after one day.

We note some properties of the transition matrix:

* tij represents the entry in row i column j
* tij = the probability of moving from state represented by row i to the state represented by row j in a single transition
* tij is a conditional probability which we can write as:   
   tij = P(next state is the state in column j | current state is the state in row i)
* Each row adds to 1
* All entries are between 0 and 1, inclusive because they are probabilities.
* The transition matrix represents change over one transition period; in this example one transition is a fixed unit of time of one day.

***Example 1*** A city is served by two cable TV companies, BestTV and CableCast.

* Due to their aggressive sales tactics, each year 40% of BestTV customers switch to CableCast; the other 60% of BestTV customers stay with BestTV.
* On the other hand, 30% of the CableCast customers switch to Best TV.

The two states in this example are BestTV and CableCast. Express the information above as a transition matrix which displays the probabilities of going from one state into another state.

***Solution:*** The transition matrix is:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **Next year** | |  |  | |
|  |  |  | BestTV | CableCast | | |  | |
| **This year** | BestTV |  | .60 | .40 | | |  | |
|  | CableCast |  | .30 | .70 | | |  | |

As previously noted, the reader should observe that a transition matrix is always a square matrix because all possible states must have both rows and columns. All entries in a transition matrix are non-negative as they represent probabilities. And, since all possible outcomes are considered in the Markov process, the sum of the row entries is always 1.

With a larger transition matrix, the ideas in Example 1 could be expanded to represent a market with more than 2 cable TV companies. The concepts of brand loyalty and switching between brands demonstrated in the cable TV example apply to many types of products, such as cell phone carriers, brands of regular purchases such as food or laundry detergent, brands major purchases such as cars or appliances, airlines that travelers choose when booking flights, or hotels chains that travelers choose to stay in.

The transition matrix shows the probabilities for transitions between states at two consecutive times. We need a way to represent the distribution among the states at a particular point in time. To do this we use a row matrix called a **state vector**. The state vector is a row matrix that has only one row; it has one column for each state. The entries show the distribution by state at a given point in time. All entries are between 0 and 1 inclusive, and the sum of the entries is 1.

For the bike share example with 3 bike share stations, the state vector is a 1x3 matrix with   
1 row and 3 columns. Suppose that when we start observing our bike share program,   
30% of the bikes are at station A, 45% of the bikes are at station B, and 25% are at station C. The initial state vector is



The subscript 0 indicates that this is the initial distribution, before any transitions occur.

If we want to determine the distribution after one transition, we’ll need to find a new state vector that we’ll call V1. The subscript 1 indicates this is the distribution after 1 transition has occurred.

We find V1 by multiplying V0 by the transitionmatrix T, as follows:

V1= V0T 

After 1 day (1 transition), 16 % of the bikes are at station A, 44.5 % are at station B and 39.5% are at station C.

We showed the step by step work for the matrix multiplication above. In the future we’ll generally use technology, such as the matrix capabilities of our calculator, to perform any necessary matrix multiplications, rather than showing the step by step work

Suppose now that we want to know the distribution of bicycles at the stations after two days. We need to find V2, the state vector after two transitions. To find V2 , we multiply the state vector after one transition V1 by the transition matrix T.

V2= V1T 

We note that V1= V0T, so V2= V1T = (V0T)T = V0T2

This gives an equivalent method to calculate the distribution of bicycles on day 2:



V2=V0T2 =

After 2 days (2 transitions), 13.2 % of the bikes are at station A, 38.65 % are at station B   
and 48.15% are at station C.

We need to examine the following: What is the meaning of the entries in the matrix T2?



The entries in T2 tell us the probability of a bike being at a particular station after two transitions, given its initial station.

* Entry t13 in row 1 column 3 tells us that a bike that is initially borrowed from station A has a probability of 0.37 of being in station C after two transitions.
* Entry t32 in row 3 column 2 tells us that a bike that is initially borrowed from station C has a probability of 0. 19 of being in station B after two transitions.

Similarly, if we raise transition matrix T to the nth power, the entries in Tn tells us the probability of a bike being at a particular station after n transitions, given its initial station.

And if we multiply the initial state vector V0 by Tn, the resulting row matrix Vn=V0Tn is the distribution of bicycles after n transitions.

***Example 2*** Refer to Example 1 for the transition matrix for market shares for subscribers to two cable TV companies.a. Suppose that today 1/4 of customers subscribe to BestTV and 3/4 of customers subscribe to CableCast. After 1 year, what percent subscribe to each company?

b. Suppose instead that today of 80% of customers subscribe to BestTV and 20% subscribe to CableCast. After 1 year, what percent subscribe to each company?

***Solution:*** a.The initial distribution given by the initial state vector is a 1×2 matrix



and the transition matrix is



After 1 year, the distribution of customers is



After 1 year, 37.5% of customers subscribe to BestTV and 62.5% to CableCast.

b. The initial distribution given by the initial state vector . Then



In this case, after 1 year, 54% of customers subscribe to BestTV and 46% to CableCast.

Note that the distribution after one transition depends on the initial distribution; the distributions in parts (a) and (b) are different because of the different initial state vectors.

***Example 3*** Professor Symons either walks to school, or he rides his bicycle. If he walks to school one day, then the next day, he will walk or cycle with equal probability. But if he bicycles one day, then the probability that he will walk the next day is 1/4. Express this information in a transition matrix.

***Solution:*** We obtain the following transition matrix by properly placing the row and column entries. Note that if, for example, Professor Symons bicycles one day, then the probability that he will walk the next day is 1/4, and therefore, the probability that he will bicycle the next day is 3/4.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | | **Next Day** | | | |  | | | |  | |
|  |  |  | | Walk | | Bicycle | | |  | |
| **Initial Day** | Walk | |  | | 1/2 | | 1/2 | | |  | | |
|  | Bicycle | |  | | 1/4 | | 3/4 | | |  | | |

***Example 4*** In Example 3, if it is assumed that the initial day is Monday, write a matrix that gives probabilities of a transition from Monday to Wednesday.

***Solution:*** If today is Monday, then Wednesday is two days from now, representing two transitions. We need to find the square, T2, of the original transition matrix T, using matrix multiplication.

T =

T2 = T×T =

=

=

Recall that we do not obtain T2 by squaring each entry in matrix T, but obtain T2 by multiplying matrix T by itself using matrix multiplication.

We represent the results in the following matrix.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **Wednesday** | |  | | |  |
|  |  |  | Walk | Bicycle | | | |  |
| **Monday** | Walk |  | 3/8 | 5/8 | |  |
|  | Bicycle |  | 5/16 | 11/16 | |  |

We interpret the probabilities from the matrix T2 as follows:

P(Walked Wednesday | Walked Monday) = 3/8.

P(Bicycled Wednesday | Walked Monday) = 5/8.

P(Walked Wednesday | Bicycled Monday) = 5/16.

P(Bicycled Wednesday | Bicycled Monday) = 11/16.

***Example 5*** The transition matrix for Example 3 is given below.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **Tuesday** | |  | | | | |  |
|  |  |  | Walk | Bicycle | | |  | |
| **Monday** | Walk |  | 1/2 | 1/2 | |  | |
|  | Bicycle |  | 1/4 | 3/4 | |  | |

Write the transition matrix from a) Monday to Thursday, b) Monday to Friday.

***Solution:*** In writing a transition matrix from Monday to Thursday, we are moving from one state to another in three steps. That is, we need to compute T3.

T3 =

b) To find the transition matrix from Monday to Friday, we are moving from one state to another in 4 steps. Therefore, we compute T4.

T4 =

It is important that the student is able to interpret the above matrix correctly. For example, the entry 85/128, states that if Professor Symons walked to school on Monday, then there is 85/128 probability that he will bicycle to school on Friday.

There are certain Markov chains that tend to stabilize in the long run. We will examine these more deeply later in this chapter. The transition matrix we have used in the above example is just such a Markov chain. The next example deals with the long term trend or steady-state situation for that matrix.

***Example 6*** Suppose Professor Symons continues to walk and bicycle according to the transition matrix given in Example 3. In the long run, how often will he walk to school, and how often will he bicycle?

***Solution:*** If we examine higher powers of the transition matrix T, we will find that it stabilizes.

T5 = T10 =

And T20 = and Tn = for n > 20

The matrix shows that in the long run, Professor Symons will walk to school 1/3 of the time and bicycle 2/3 of the time.

When this happens, we say that the system is in steady-state or state of equilibrium. In this situation, all row vectors are equal. If the original matrix is an n by n matrix, we get n row vectors that are all the same. We call this vector a **fixed probability vector** or the **equilibrium vector** E. In the above problem, the fixed probability vector E is [1/3 2/3]. Furthermore, if the equilibrium vector E is multiplied by the original matrix T, the result is the equilibrium vector E. That is,

ET = E , or =

# 10.2 Applications of Markov Chains

In this section you will examine some ways in which Markov Chains models are used in business, finance, public health and other fields of application

In the previous section, we examined several applications of Markov chains. Before we proceed further in our investigations of the mathematics of Markov chains in the next sections, we take the time in this section to examine how Markov chains are used in real world applications.

In our bike share program example, we modelled the distribution of the locations of bicycles at bike share stations using a Markov chain. Markov chains have been proposed to model locations of cars distributed among multiple car rental locations for a car rental company, and locations of cars in car share programs. Markov chains models analyze package delivery schedules when packages are transported between several intermediate transport and storage locations on their way to their final destination. In these situations, Markov chains are often one part of larger mathematical models using a combination of other techniques, such as optimization to maximize profit or revenue or minimize cost using linear programming.

In our cable TV example, we modelled market share in a simple example of two cable TV providers. Markov chains can be similarly used in market research studies for many types of products and services, to model brand loyalty and brand transitions as we did in the cable TV model. In the field of finance, Markov chains can model investment return and risk for various types of investments.

Markov chains can model the probabilities of claims for insurance, such as life insurance and disability insurance, and for pensions and annuities. For example, for disability insurance, a much simplified model might include states of healthy, temporarily disabled, permanently disabled, recovered, and deceased; additional refinements could distinguish between disabled policyholders still in the waiting period before collecting benefits and claims actively collecting benefits.

Markov chains have been used in the fields of public health and medicine. Markov chains models of HIV and AIDS include states to model HIV transmission, progression to AIDs, and survival (living with HIV or AIDS) versus death due to AIDS. Comparing Markov chain models of HIV transmission and AIDs progression for various risk groups and ethnic groups can guide public health organizations in developing strategies for reducing risk and managing care for these various groups of people. In general, modeling transmission of various infectious diseases with Markov chains can help in determination of appropriate public health responses to monitor and slow or halt the transmission of these diseases and to determine the most efficient ways to approach treating the disease.

Markov chains have many health applications besides modeling spread and progression of infectious diseases. When analyzing infertility treatments, Markov chains can model the probability of successful pregnancy as a result of a sequence of infertility treatments. Another medical application is analysis of medical risk, such as the role of risk in patient condition following surgery; the Markov chain model quantifies the probabilities of patients progressing between various states of health.

Markov chains are used in ranking of websites in web searches. Markov chains model the probabilities of linking to a list of sites from other sites on that list; a link represents a transition. The Markov chain is analyzed to determine if there is a steady state distribution, or equilibrium, after many transitions. Once equilibrium is identified, the pages with the probabilities in the equilibrium distribution determine the ranking of the webpages. This is a very simplified description of how Google uses Markov chains and matrices to determine “Page rankings” as part of their search algorithms.

Of course, a real world use of such a model by Google would involve immense matrices with thousands of rows and columns. The size of such matrices requires some modifications and use of more sophisticated techniques than we study for Markov chains in this course. However the methods we study form the underlying basis for this concept. It is interesting to note that the term Page ranking does not refer to the fact that webpages are ranked, but instead is named after Google founder Larry Page, who was instrumental in developing this application of Markov chains in the area of web page search and rankings.

Markov chains are also used in quality analysis of cell phone and other communications transmissions. Transition matrices model the probabilities of certain types of signals being transmitted in sequence. Certain sequences of signals are more common and expected, having higher probabilities; on the other hand, other sequences of signals are rare and have low probabilities of occurrence. If certain sequences of signals that are unlikely to occur actually do occur, that might be an indication of errors in transmissions; Markov chains help identify the sequences that represent likely transmission errors.

# 10.3 Regular Markov Chains

In this section, you will learn to:

1. identify Regular Markov Chains, which have an equilibrium or steady state in the long run

2. find the long term equilibrium for a Regular Markov Chain.

At the end of section 10.1, we examined the transition matrix T for Professor Symons walking and biking to work. As we calculated higher and higher powers of T, the matrix started to stabilize, and finally it reached its **steady-state** or **state of equilibrium**. When that happened, all the row vectors became the same, and we called one such row vector a **fixed probability vector** or an **equilibrium vector** E. Furthermore, we discovered that ET = E.

In this section, we wish to answer the following four questions.

1) Does every Markov chain reach a state of equilibrium? Is there a way to determine if a Markov chain reaches a state of equilibrium?

2) Does the product of an equilibrium vector and its transition matrix always equal the equilibrium vector? That is, does ET = E?

3) Can the equilibrium vector E be found without raising the matrix to higher powers?

4) Does the long term market share distribution for a Markov chain depend on the initial market share?

***Question 1*** Does every Markov chain reach the state of equilibrium?

***Answer:*** No. Some Markov chains reach a state of equilibrium but some do not. Some Markov chains transitions do not settle down to a fixed or equilibrium pattern. Therefore we’d like to have a way to identify Markov chains that do reach a state of equilibrium.

One type of Markov chains that do reach a state of equilibrium are called **regular** Markov chains. A Markov chain is said to be a **regular Markov chain** if some power of its transition matrix T has only positive entries.

To determine if a Markov chain is regular, we examine its transition matrix T and powers, Tn, of the transition matrix. If we find any power n for which Tn has only positive entries (no zero entries), then we know the Markov chain is regular and is guaranteed to reach a state of equilibrium in the long run.

Fortunately, we don’t have to examine too many powers of the transition matrix T to determine if a Markov chain is regular; we use technology, calculators or computers, to do the calculations. There is a theorem that says that if an nxn transition matrix represents n states, then we need only examine powers Tm up to m = ( n−1)2 + 1.   
If some power of the transition matrix Tm is going to have only positive entries, then that will occur for some power m ≤ ( n−1)2 + 1

For example, if T is a 3x3 transition matrix, then m = ( n−1)2 + 1= ( 3−1)2 + 1=5.

* If we examine T, T2, T3, T4 and T5, and find that any of those matrices has only positive entries, then we know T is regular.
* If however, T, T2, T3, T4 and T5 all have at least one zero entry and none of them have all positive entries, then we can stop checking. All higher powers of T will also have at least one zero entry, and T will not be regular.

***Example 1*** Determine whether the following Markov chains are regular.

a. A = b. B =

***Solution:*** a). The transition matrix A does not have all positive entries. But it is a regular Markov chain because

A2 =

has only positive entries.

b.The matrix B is not a regular Markov chain because every power of B has an entry 0 in the first row, second column position.

B = and B2 = = 

Since B is a 2x2 matrix, m = (2−1)2+1= 2. We’ve examined B and B2, and discovered that neither has all positive entries. We don’t need to examine any higher powers of B; B is not a regular Markov chain.

In fact, we can show that all 2 by 2 matrices that have a zero in the first row, second column position are not regular. Consider the following matrix M.

M =

M2 = =

Observe that the first row, second column entry, a . 0 + 0 . c, will always be zero, regardless of what power we raise the matrix to.

***Question 2*** Does the product of an equilibrium vector and its transition matrix always equal the equilibrium vector? That is, does ET = E?

***Answer:*** At this point, the reader may have already guessed that the answer is yes if the transition matrix is a regular Markov chain. We try to illustrate with the following example from section 10.1.

A city is served by two cable TV companies, BestTV and CableCast. Due to their aggressive sales tactics, each year 40% of BestTV customers switch to CableCast; the other 60% of BestTV customers stay with BestTV. On the other hand, 30% of the CableCast customers switch to Best RV and 70% of CableCast customers stay with CableCast.

The transition matrix is given below.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **Next Year** | |  |  | |
|  |  |  | BestTV | CableCast | | |  | |
| **Initial Year** | BestTV |  | .60 | .40 | | |  | |
|  | CableCast |  | .30 | .70 | | |  | |

If the initial market share for BestTV is 20% and for CableCast is 80%, we'd like to know the long term market share for each company.

Let matrix T denote the transition matrix for this Markov chain, and V0 denote the matrix that represents the initial market share. Then V0 and T are as follows:

V0 = and T =

Since each year people switch according to the transition matrix T, after one year the distribution for each company is as follows:

V1= V0 T = =

After two years, the market share for each company is

V2= V1 T = =

After three years the distribution is

V3= V2 T = =

After 20 years the market share are given by V20 = V0 T20 = .

After 21 years,V21 = V0 T21 = ; market shares are stable and did not change.

The market share after 20 years has stabilized to . This means that

=

Once the market share reaches an equilibrium state, it stays the same, that is, ET = E.

This helps us answer the next question.

***Question 3*** Can the equilibrium vector E be found without raising the transition matrix T to large powers?

***Answer:*** The answer to the second question provides us with a way to find the equilibrium vector E.

The answer lies in the fact that ET = E.

Since we have the matrix T, we can determine E from the statement ET = E.

Suppose E =, then ET = E gives us

=

=

=

.30e + .30 = e

e = 3/7

Therefore, E =

As a result of our work in questions 2 and 3, we see that we have a choice of methods to find the equilibrium vector.

**Method 1: We can determine if the transition matrix T is regular. If T is regular, we know there is an equilibrium and we can use technology to find a high power of T.**

* For the question of what is a sufficiently high power of T, there is no “exact” answer.
* Select a “high power”, such as n=30, or n=50, or n=98. Evaluate Tn on your calculator or with a computer. Check if Tn+1 = Tn. If Tn+1 = Tn and all the rows of Tn are the same, then we’ve found the equilibrium. The equilibrium vector is one row of Tn. But if you did not find equilibrium yet for a regular Markov chain, try using a higher power of T.

**Method 2: We can solve the matrix equation ET=E.**

* The disadvantage of this method is that it is a bit harder, especially if the transition matrix is larger than 2x2. However it’s not as hard as it seems, if T is not too large a matrix, because we can use the methods we learned in chapter 2 to solve the system of linear equations, rather than doing the algebra by hand.
* The advantage of solving ET = E as in Method 2 is that it can be used with matrices that are not regular. If a matrix is regular, it is guaranteed to have an equilibrium solution.   
  If a matrix is not regular, then it may or may not have an equilibrium solution, and solving ET = E will allow us to prove that it has an equilibrium solution even if the matrix is not regular.   
  (In mathematics we say that being a regular matrix is a “sufficient” condition for having an equilibrium, but is not a necessary condition.)

***Question 4*** Does the long term market share for a Markov chain depend on the initial market share?

***Answer:*** We will show that the final market share distribution for a Markov chain does not depend upon the initial market share. In fact, one does not even need to know the initial market share distribution to find the long term distribution.

Furthermore, the final market share distribution can be found by simply raising the transition matrix to higher powers.

Consider the initial market share V0=, and the transition matrix

T = for BestTV and CableCast in the above example.   
Recall we found Tn, for very large n, to be .

Using our calculators, we can easily verify that for sufficiently large n (we used n = 30), V0Tn = =

No matter what the initial market share, the product is .

If instead the initial share is W0= , then for sufficiently large n

W0Tn = =

For any distribution A = ­, for example,

=   
=

It makes sense; the entry 3/7(a) + 3/7(1 – a), for example, will always equal 3/7.

***Example 2*** Three companies, A, B, and C, compete against each other. The transition matrix T for people switching each month among them is given by the following transition matrix.

Next Month

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | | Company A | Company B | Company C |  | |
|  | Company A | |  | .1 | .3 | .6 | |  | |
| **Initial Month** | Company B | |  | .6 | .2 | .2 | |  | |
|  | Company C | |  | .1 | .3 | .6 | |  | |

If the initial market share for the companies A, B, and C is , what is the long term distribution?

***Solution:*** Since the long term market share does not depend on the initial market share, we can simply raise the transition market share to a large power and get the distribution.

T20 =

In the long term, Company A has 13/55 (about 23.64%) of the market share, Company B has 3/11 (about 27.27%) of the market share, and Company C has 27/55 (about 49.09%) of the market share.

We summarize as follows:

|  |
| --- |
| **Regular Markov Chains**  A Markov chain is said to be a Regular Markov chain if some power of it has only positive entries.  Let T be a transition matrix for a regular Markov chain.  1. As we take higher powers of T, Tn, as n becomes large, approaches a state of equilibrium.  2. If V0 is any distribution vector, and E an equilibrium vector,  then V0Tn = E.  3. Each row of the equilibrium matrix Tn is a unique equilibrium vector E such that ET = E.  4. The equilibrium distribution vector E can be found by letting ET = E. |

# 10.4 Absorbing Markov Chains

In this section you will learn to:

1. identify absorbing states and absorbing Markov chains

2. solve and interpret absorbing Markov chains.

In this section, we will study a type of Markov chain in which when a certain state is reached, it is impossible to leave that state. Such states are called **absorbing states**, and a Markov Chain that has at least one such state is called an **Absorbing Markov chain.**Suppose you have the following transition matrix.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | S1 | S2 | S3 |  |
| S1 |  | .1 | .3 | .6 |  |
| S2 |  | 0 | 1 | 0 |  |
| S3 |  | .3 | .2 | .5 |  |

The state S2 is an absorbing state, because the probability of moving from state S2 to state S2 is 1. Which is another way of saying that if you are in state S2, you will remain in state S2.

In fact, this is the way to identify an absorbing state. If the probability in row i and column i , pii, is 1, then state Si is an absorbing state.

***Example 1*** Consider transition matrices A, B, C for Markov chains shown below.Which of the following Markov chains have an absorbing state?



***Solution:***

 has S2 as an absorbing state.   
If we are in state S2, we can not leave it.

From state S2, we can not transition to state S1 or S3; the probabilities are 0.   
The probability of transition from state S2 to state S2 is 1.

 does not have any absorbing states.   
From state S1, we always transition to state S2. From state S2 we always transition to state S3. From state S3 we always transition to state S1. In this matrix, it is never possible to stay in the same state during a transition.

 has two absorbing states, S3 and S4.

From state S3, you can only remain in state S3, and never transition to any other states. Similarly from state S4, you can only remain in state S4, and never transition to any other states.

We summarize how to identify absorbing states.

A state S is an absorbing state in a Markov chain in the transition matrix if

* The row for state S has one 1 and all other entries are 0

AND

* The entry that is 1 is on the main diagonal (row = column for that entry), indicating that we can never leave that state once it is entered.

Next we define an absorbing Markov Chain

A Markov chain is an absorbing Markov Chain if

* It has at least one absorbing state

AND

* From any non-absorbing state in the Markov chain, it is possible to eventually move to some absorbing state (in one or more transitions).

***Example 2*** Consider transition matrices C and D for Markov chains shown below.Which of the following Markov chains is an absorbing Markov Chain?



***Solution:*** C is an absorbing Markov Chain but D is not an absorbing Markov chain.

Matrix C has two absorbing states, S3 and S4, and it is possible to get to state S3 and S4 from S1 and S2.

Matrix D is not an absorbing Markov chain.has two absorbing states, S1 and S2, but it is never possible to get to either of those absorbing states from either S4 or S5. If you are in state S4 or S5, you always remain transitioning between states S4 or S5m and can never get absorbed into either state S1 or S2

In the remainder of this section, we’ll examine absorbing Markov chains with two classic problems: the random drunkard’s walk problem and the gambler's ruin problem. And finally we’ll conclude with an absorbing Markov model applied to a real world situation.

## DRUNKARD’S RANDOM WALK

In this example we briefly examine the basic ideas concerning absorbing Markov chains.

***Example 3*** A man walks along a three block portion of Main St. His house is at one end of the three block section. A bar is at the other end of the three block section. Each time he reaches a corner he randomly either goes forward one block or turns around and goes back one block. If he reaches home or the bar, he stays there. The four states are Home (H), Corner 1(C1), Corner 2 (C2) and Bar (B).  
Write the transition matrix and identify the absorbing states. Find the probabilities of ending up in each absorbing state depending on the initial state.

***Solution:*** The transition matrix is written below.



Home and the Bar are absorbing states. If the man arrives home, he does not leave.   
If the man arrives at the bar, he does not leave. Since it is possible to reach home or the bar from each of the other two corners on his walk, this is an absorbing Markov chain.

We can raise the transition matrix T to a high power, n. One we find a power Tn that remains stable, it will tell us the probability of ending up in each absorbing state depending on the initial state.

 and 

T91 = T90; the matrix does not change as we continue to examine higher powers.   
We see that in the long-run, the Markov chain must end up in an absorbing state.   
In the long run, the man must eventually end up at either his home or the bar.

The second row tells us that if the man is at corner C1, then there is a 2/3 chance he will end up at home and a 1/3 chance he will end up at the bar.

The third row tells us that if the man is at corner C2, then there is a 1/3 chance he will end up at home and a 2/3 chance he will end up at the bar.

Once he reaches home or the bar, he never leaves that absorbing state.

Note that while the matrix Tn for sufficiently large n has become stable and is not changing, it does not represent an equilibrium matrix. The rows are not all identical, as we found in the regular Markov chains that reached an equilibrium.

We can write a smaller “solution matrix” by retaining only rows that relate to the non-absorbing states and retaining only the columns that relate to the absorbing states.

Then the solution matrix will have rows C1 and C2, and columns H and B.  
The solution matrix is

Solution Matrix: 

The first row of the solution matrix shows that if the man is at corner C1, then there is a 2/3 chance he will end up at home and a 1/3 chance he will end up at the bar.

The second row of the solution matrix shows that if the man is at corner C2, then there is a 1/3 chance he will end up at home and a 2/3 chance he will end up at the bar.

The solution matrix does not show that eventually there is 0 probability of ending up in C1 or C2, or that if you start in an absorbing state H or B, you stay there. The smaller solution matrix assumes that we understand these outcomes and does not include that information.

The next example is another classic example of an absorbing Markov chain. In the next example we examine more of the mathematical details behind the concept of the solution matrix.

## GAMBLER'S RUIN PROBLEM

***Example 4*** A gambler has $3,000, and she decides to gamble $1,000 at a time at a Black Jack table in a casino in Las Vegas. She has told herself that she will continue playing until she goes broke or has $5,000. Her probability of winning at Black Jack is .40. Write the transition matrix, identify the absorbing states, find the solution matrix, and determine the probability that the gambler will be financially ruined at a stage when she has $2,000.

***Solution:*** The transition matrix is written below. Clearly the state 0 and state 5K are the absorbing states. This makes sense because as soon as the gambler reaches 0, she is financially ruined and the game is over. Similarly, if the gambler reaches $5,000, she has promised herself to quit and, again, the game is over. The reader should note that p00 = 1, and p55 = 1.

Further observe that since the gambler bets only $1,000 at a time, she can raise or lower her money only by $1,000 at a time. In other words, if she has $2,000 now, after the next bet she can have $3,000 with a probability of .40 and $1,000 with a probability of .60.

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | 0 | 1K | | 2K | 3K | | 4K | 5K |  | |
| 0 |  | 1 | 0 | 0 | | 0 | 0 | | 0 | |  | |
| 1K |  | .60 | 0 | .40 | | 0 | 0 | | 0 | |  | |
| 2K |  | 0 | .60 | 0 | | .40 | 0 | | 0 | |  | |
| 3K |  | 0 | 0 | .60 | | 0 | .40 | | 0 | |  | |
| 4K |  | 0 | 0 | 0 | | .60 | 0 | | .40 | |  | |
| 5K |  | 0 | 0 | 0 | | 0 | 0 | | 1 | |  | |

To determine the long term trend, we raise the matrix to higher powers until all the non-absorbing states are absorbed. This is the called the **solution matrix.**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | 0 | 1K | 2K | 3K | 4K | 5K |  |
| 0 |  | 1 | 0 | 0 | 0 | 0 | 0 |  |
| 1K |  | 195/211 | 0 | 0 | 0 | 0 | 16/211 |  |
| 2K |  | 171/211 | 0 | 0 | 0 | 0 | 40/211 |  |
| 3K |  | 135/211 | 0 | 0 | 0 | 0 | 76/211 |  |
| 4K |  | 81/211 | 0 | 0 | 0 | 0 | 130/211 |  |
| 5K |  | 0 | 0 | 0 | 0 | 0 | 1 |  |

The solution matrix is often written in the following form, where the non-absorbing states are written as rows on the side, and the absorbing states as columns on the top.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | 0 | 5K |  |
| 1K |  | 195/211 | 16/211 |  |
| 2K |  | 171/211 | 40/211 |  |
| 3K |  | 135/211 | 76/211 |  |
| 4K |  | 81/211 | 130/211 |  |

The table lists the probabilities of getting absorbed in state 0 or state 5K starting from any of the four non-absorbing states. For example, if at any instance the gambler has $3,000, then her probability of financial ruin is 135/211 and her probability reaching 5K is 76/211.

***Example 5*** Solve the Gambler's Ruin Problem of Example 4 without raising the matrix to higher powers, and determine the number of bets the gambler makes before the game is over.

***Solution:*** In solving absorbing states, it is often convenient to rearrange the matrix so that the rows and columns corresponding to the absorbing states are listed first. This is called the **Canonical form.** The transition matrix of Example 1 in the canonical form is listed below.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | | 0 | 5K | 1K | | 2K | 3K | 4K | |  | |
| 0 |  | 1 | | 0 | 0 | 0 | | 0 | | 0 | |  | |
| 5K |  | 0 | | 1 | 0 | 0 | | 0 | | 0 | |  | |
| 1K |  | .60 | | 0 | 0 | .40 | | 0 | | 0 | |  | |
| 2K |  | 0 | | 0 | .60 | 0 | | .40 | | 0 | |  | |
| 3K |  | 0 | | 0 | 0 | .60 | | 0 | | .40 | |  | |
| 4K |  | 0 | | .40 | 0 | 0 | | .60 | | 0 | |  | |

The canonical form divides the transition matrix into four sub-matrices as listed below.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | Absorbing | Non-absorbing |  |
| Absorbing states |  | In | O |  |
| Non-absorbing states |  | A | B |  |

The matrix F = (In– B)–1 is called the fundamental matrix for the absorbing Markov chain, where In is an identity matrix of the same size as B. The i, j-th entry of this matrix tells us the average number of times the process is in the non-absorbing state j before absorption if it started in the non-absorbing state i.

The matrix F = (In– B)–1 for our problem is listed below.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | 1K | 2K | 3K | 4K |  |
|  | 1K |  | 1.54 | .90 | .47 | .19 |  |
| F = | 2K |  | 1.35 | 2.25 | 1.18 | .47 |  |
|  | 3K |  | 1.07 | 1.78 | 2.25 | .90 |  |
|  | 4K |  | .64 | 1.07 | 1.35 | 1.54 |  |

You can use your calculator, or a computer, to calculate matrix F.

The Fundamental matrix F helps us determine the average number of games played before absorption.

According to the matrix, the entry 1.78 in the row 3, column 2 position says that the gambler will play the game 1.78 times before she goes from $3K to $2K. The entry 2.25 in row 3, column 3 says that if the gambler now has $3K, she will have $3K on the average 2.25 times before the game is over.

We now address the question of how many bets will she have to make before she is absorbed, if the gambler begins with $3K?

If we add the number of games the gambler plays in each non-absorbing state, we get the average number of games before absorption from that state. Therefore, if the gambler starts with $3K, the average number of Black Jack games she will play before absorption is

1.07 + 1.78 + 2.25 + .90 = 6.0

That is, we expect the gambler will either have $5,000 or nothing on the 7th bet.

Lastly, we find the solution matrix without raising the transition matrix to higher powers. The matrix FA gives us the solution matrix.

FA = = 

which is the same as the following matrix we obtained by raising the transition matrix to higher powers.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | 0 | 5K |  |
| 1K |  | 195/211 | 16/211 |  |
| 2K |  | 171/211 | 40/211 |  |
| 3K |  | 135/211 | 76/211 |  |
| 4K |  | 81/211 | 130/211 |  |

***Example 6*** At a professional school, students need to take and pass an English writing/speech class in order to get their professional degree. Students must take the class during the first quarter that they enroll. If they do not pass the class they take it again in the second semester. If they fail twice, they are not permitted to retake it again, and so would be unable to earn their degree.

Students can be in one of 4 states: passed the class (P), enrolled in the class for the first time (C), retaking the class (R) or failed twice and can not retake (F). Experience shows 70% of students taking the class for the first time pass and 80% of students taking the class for the second time pass.

Write the transition matrix and identify the absorbing states.   
Find the probability of being absorbed eventually in each of the absorbing states.

***Solution:*** The absorbing states are P (pass) and F (fail repeatedly and can not retake).  
The transition matrix T is shown below.



If we raise the transition matrix T to a high power, we find that it remains stable and gives us the long-term probabilities of ending up in each of the absorbing states.



Of students currently taking the class for the first time, 94% will eventually pass.   
6% will eventually fail twice and be unable to earn their degree.

Of students currently taking the class for the second time time, 80% will eventually pass. 20% will eventually fail twice and be unable to earn their degree.

The solution matrix contains the same information in abbreviated form



Note that in this particular problem, we don’t need to raise T to a “very high” power. If we find T2, we see that it is actually equal to Tn for higher powers n. Tn becomes stable after two transitions; this makes sense in this problem because after taking the class twice, the student must have passed or is not permitted to retake it any longer. Therefore the probabilities should not change any more after two transitions; by the end of two transitions, every student has reached an absorbing state.

|  |
| --- |
| **Absorbing Markov Chains**  1. A Markov chain is an absorbing Markov chain if it has at least one absorbing state. A state i is an absorbing state if once the system reaches state i, it stays in that state; that is, pii = 1.  2. If a transition matrix T for an absorbing Markov chain is raised to higher powers, it reaches an absorbing state called the solution matrix and stays there. The i, j-th entry of this matrix gives the probability of absorption in state j while starting in state i.  3. Alternately, the solution matrix can be found in the following manner.  a. Express the transition matrix in the canonical form as below.  T =  where In is an identity matrix, and **0** is a matrix of all zeros.  b. The fundamental matrix F = (I – B)–1. The fundamental matrix helps us find the number of games played before absorption.  c. FA is the solution matrix, whose i, j-th entry gives the probability of absorption in state j while starting in state i.  4. The sum of the entries of a row of the fundamental matrix gives us the expected number of steps before absorption for the non-absorbing state associated with that row. |