HYPERBOLIC FUNCTIONS SUPPLEMENT (FOR MATH 43 ONLY)

The hyperbolic functions are based on exponential functions,
and are algebraically similar to, yet subtly different from, trigonometric functions.

In order to complete this worksheet, you need to refer back to topics from trigonometry and precalculus.
In particular, you may need to refresh yourself on the following:

Trigonometry:

quotient, reciprocal, Pythagorean, sum/difference of angles & double angle identities
(you only need to know the identities, not how to prove them)

Precalculus:

determining if a function has odd/even symmetry algebraically and graphically
determining if a function is one-to-one graphically
relationship between the graph/domain/range of a function and its inverse
using function composition to determine if two functions are inverses of each other
solving for the inverse of a function algebraically
identifying asymptotes from a graph

$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$

HYPERBOLIC FUNCTIONS

$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$

[0] [a] Find $\sinh 0$ and $\cosh 0$. Compare to $\sin 0$ and $\cos 0$.

[b] Find the domains of $\sinh x$ and $\cosh x$ using the exponential definitions.

[c] Solve $\sinh x = 0$ and $\cosh x = 0$ using the exponential definitions.

[d] Determine the signs of $\sinh x$ and $\cosh x$ if $x > 0$, and if $x < 0$.

[e] Determine the size and sign of $\sinh x$ and $\cosh x$ as $x \to -\infty$, and as $x \to \infty$, using the size and sign of e^x and e^{-x} as $x \to -\infty$, and as $x \to \infty$.

[1] Rewrite in terms of exponential functions using the definitions above, simplify, then rewrite in terms of hyperbolic functions, if possible.

[a] $\sinh(-x)$ [b] $\cosh(-x)$

[c] $\cosh^2 x + \sinh^2 x$ [d] $\cosh^2 x - \sinh^2 x$

[e] $2\sinh x \cosh x$ [f] $e^x \sinh x$

[g] $\dfrac{\cosh x}{e^x}$ [h] $\cosh x + \sinh x$

[i] $\cosh x - \sinh x$ [j] $\sinh(\ln x)$

[k] $\cosh(2\ln x)$ [l] $\cosh(\ln 3)$

[m] $\sinh(3\ln 2)$ [n] $\cosh(\frac{1}{2}\ln 5)$

[2] Based on the similarities to the trigonometric identities noted in **[1]**, guess and prove formulae for the following in terms of hyperbolic functions.

[a] $\sinh(x + y)$ [b] $\sinh(x - y)$

[c] $\cosh(x + y)$ [d] $\cosh(x - y)$

[3] Define $\tanh x = \dfrac{\sinh x}{\cosh x}$.

[a] Rewrite $\tanh x$ in terms of exponential functions.

[b] Find $\tanh 0$, find the domain of $\tanh x$, and solve $\tanh x = 0$ in two ways

[i] by using the original definition above
[ii] by using your answer in **[3][a]**

[c] Simplify $\tanh(-x)$ in two ways

[i] by first rewriting in terms of exponential functions, simplifying, then rewriting in terms of hyperbolic functions
[ii] by using your answers in **[1][a]** and **[1][b]**

[d] Define the three remaining hyperbolic functions in a parallel fashion in terms of other hyperbolic functions.

[e] Repeat **[3][a]**, **[3][b][i]** and **[3][c][ii]** for the three remaining hyperbolic functions.

[f] Rewrite in terms of exponential functions using the definitions above, simplify, then rewrite in terms of hyperbolic functions, if possible.

[i] $\coth(\ln 2)$

[ii] $\operatorname{sech}(2\ln 3)$

[iii] $\tanh(-\ln 5)$

[iv] $\operatorname{csch}(\frac{1}{2}\ln 7)$

Once you get used to the identities, it is much easier to manipulate the hyperbolic functions without rewriting them in terms of exponential functions.

[4] You should have discovered a hyperbolic parallel to the Pythagorean Identity in **[1][d]**.

[a] Rewrite the identity in **[1][d]** in two ways

[i] by solving for $\sinh^2 x$
[ii] by solving for $\cosh^2 x$

[b] Rewrite the identity in **[1][c]** in two ways

[i] by substituting using your answer from **[4][a][i]**
[ii] by substituting using your answer from **[4][a][ii]**

[c] Find the hyperbolic parallels to the other Pythagorean Identities by dividing both sides of **[1][d]**

[i] by $\sinh^2 x$
[ii] by $\cosh^2 x$

[d] If $\sinh x = \dfrac{1}{2}$, find the values of the other hyperbolic functions using the definitions and identities above.

[e] If $\coth x = 2$, find the values of the other hyperbolic functions using the definitions and identities above.

[f] If $\operatorname{sech} x = \dfrac{1}{2}$ and $x < 0$, find the values of the other hyperbolic functions using the definitions and identities above.

INVERSE HYPERBOLIC FUNCTIONS

[5] [a] Find all intercepts of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$ using your answers in **[0][a][c]** and **[3][b]**.

[b] Determine which quadrants the graphs of $y = \sinh x$ and $y = \cosh x$ should lie in
using your answers in **[0][d]**.

[c] Determine the sign of $\tanh x$ if $x > 0$, and if $x < 0$,
using your answers in **[0][d]** and the definition in **[3]**,
then determine which quadrants the graph of $y = \tanh x$ should lie in.

[d] Describe the symmetry of the graphs of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$ using your answers in **[1][a][b]** and **[3][c]**.

[e] Describe the long run behavior of the graphs of $y = \sinh x$ and $y = \cosh x$ using your answers in **[0][e]**.

[f] Guess the graphs of $y = \sinh x$ and $y = \cosh x$ using your answers in **[5][a][b][d][e]**.
Then use your calculator's graphing capabilities to sketch $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$.
State the domain and range of each graph. State the equations of all asymptotes.

[g] Recall that a function has an inverse function if and only if the function is one-to-one.
Which of the three functions in **[5][f]** is (are) one-to-one and why ?

[h] A function which is not one-to-one, can be "made" one-to-one by restricting its domain (eg. the trigonometric functions). How would you restrict the domain of the non one-to-one function(s) in **[5][g]** to "make" it (them) one-to-one ?

[6] Solve $\sinh x = 1$ and $\cosh x = 1$ by using the exponential definitions and an algebraic substitution $z = e^x$.

[7] $y = \sinh^{-1} x$ if and only if $x = \sinh y$.

[a] Define $\cosh^{-1} x$ and $\tanh^{-1} x$.
(NOTE: A function can only have an inverse if it is one-to-one. See **[5][h]**.)

[b] **<u>Without graphing</u>**,
<u>but instead using the relationship between the graph of a function and the graph of its inverse</u>,
state the domain and range of all three inverse hyperbolic functions,
and identify all intercepts, and horizontal and vertical asymptotes.

[c] Sketch the graphs of $y = \sinh^{-1} x$, $y = \cosh^{-1} x$ and $y = \tanh^{-1} x$.

[d] Prove that $g(x) = \ln(x + \sqrt{x^2 + 1})$ is the inverse of $f(x) = \sinh x$ in three ways

[i] by simplifying $f(g(x))$ using the exponential definition of $\sinh x$
[ii] by simplifying $g(f(x))$ using the identities in **[1][h]** and **[4][a]**
[iii] by solving $x = \sinh y$ for y using the exponential definition and an algebraic substitution $z = e^y$

[e] Repeat **[7][d][iii]** to find logarithmic formulae for $\cosh^{-1} x$ and $\tanh^{-1} x$.
Then repeat **[7][d][i][ii]** to prove your formulae are correct.

ANSWERS

SOLUTIONS FOR ANSWERS MARKED ★ GIVEN IN LAST PART OF FILE

[0]

[a] $\sinh 0 = 0$
$\cosh 0 = 1$

[b] both have domain $(-\infty, \infty)$

[c] $\sinh x = 0 \quad \Rightarrow \quad x = 0$ ★
$\cosh x = 0$ has no real solution

[d] If $x > 0$, $\sinh x > 0$ ★
If $x < 0$, $\sinh x < 0$
For all real x, $\cosh x > 0$

[e] As $x \to -\infty$, $\sinh x \to -\infty$ and $\cosh x \to \infty$
As $x \to \infty$, $\sinh x \to \infty$ ★ and $\cosh x \to \infty$

[1]

[a] $-\sinh x$

[b] $\cosh x$ ★

[c] $\cosh 2x$

[d] 1

[e] $\sinh 2x$ ★

[f] $\dfrac{e^{2x} - 1}{2}$

[g] $\dfrac{1 + e^{-2x}}{2}$

[h] e^{x}

[i] e^{-x}

[j] $\dfrac{x^2 - 1}{2x}$ ★

[k] $\dfrac{x^4 + 1}{2x^2}$

[l] $\dfrac{5}{3}$

[m] $3\dfrac{15}{16}$

[n] $\dfrac{3\sqrt{5}}{5}$

[2]

[a] $\sinh x \cosh y + \cosh x \sinh y$ ★

[b] $\sinh x \cosh y - \cosh x \sinh y$

[c] $\cosh x \cosh y + \sinh x \sinh y$

[d] $\cosh x \cosh y - \sinh x \sinh y$

[3]

[a] $\dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

[b] $\tanh 0 = 0$ ★
$\tanh x$ has domain $(-\infty, \infty)$ ★
$\tanh x = 0 \quad \Rightarrow \quad x = 0$ ★

[c] $-\tanh x$ ★

[d] $\operatorname{csch} x = \dfrac{1}{\sinh x}$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$ $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{1}{\tanh x}$

[e] $\operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$

$\operatorname{csch} 0$ is undefined

$\operatorname{csch} x$ has domain $(-\infty, 0) \cup (0, \infty)$

$\operatorname{csch} x = 0$ has no real solution

$\operatorname{csch}(-x) = -\operatorname{csch} x$

$\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$\operatorname{sech} 0 = 1$

$\operatorname{sech} x$ has domain $(-\infty, \infty)$

$\operatorname{sech} x = 0$ has no real solution

$\operatorname{sech}(-x) = \operatorname{sech} x$

$\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$\coth 0$ is undefined

$\coth x$ has domain $(-\infty, 0) \cup (0, \infty)$

$\coth x = 0$ has no real solution

$\coth(-x) = -\coth x$

[f] [i] $\dfrac{5}{3}$

[ii] $\dfrac{9}{41}$ ★

[iii] $-\dfrac{12}{13}$

[iv] $\dfrac{\sqrt{7}}{3}$

[4]

[a] [i] $\sinh^2 x = \cosh^2 x - 1$

[ii] $\cosh^2 x = 1 + \sinh^2 x$

[b] [i] $2\cosh^2 x - 1$ ★

[ii] $1 + 2\sinh^2 x$

[c] [i] $\coth^2 x - 1 = \operatorname{csch}^2 x$ ★

[ii] $1 - \tanh^2 x = \operatorname{sech}^2 x$

[d] $\cosh x = \dfrac{\sqrt{5}}{2}$ ★ $\tanh x = \dfrac{\sqrt{5}}{5}$ ★ $\operatorname{csch} x = 2$

$\operatorname{sech} x = \dfrac{2\sqrt{5}}{5}$ $\coth x = \sqrt{5}$

[e] $\tanh x = \frac{1}{2}$ $\quad$ $\operatorname{sech} x = \frac{\sqrt{3}}{2}$ $\quad$ $\cosh x = \frac{2\sqrt{3}}{3}$

$\sinh x = \frac{\sqrt{3}}{3}$ $\quad$ $\operatorname{csch} x = \sqrt{3}$

[f] $\cosh x = 2$ $\quad$ $\sinh x = -\sqrt{3}$ $\quad$ $\tanh x = -\frac{\sqrt{3}}{2}$

$\operatorname{csch} x = -\frac{\sqrt{3}}{3}$ $\quad$ $\coth x = -\frac{2\sqrt{3}}{3}$

[5]

[a]

	$y = \sinh x$	$y = \cosh x$	$y = \tanh x$
$x-$intercept(s)	(0, 0)	none	(0, 0)
$y-$intercept	(0, 0)	(0, 1)	(0, 0)

[b] graph of $y = \sinh x$ lies in quadrants 1 ★ and 3

graph of $y = \cosh x$ is in quadrants 1 & 2

[c] graph of $y = \tanh x$ lies in quadrants 1 ★ and 3

[d] $y = \sinh x$ is odd and symmetric over the origin

$y = \cosh x$ is even and symmetric over the $y-$axis ★

$y = \tanh x$ is odd and symmetric over the origin

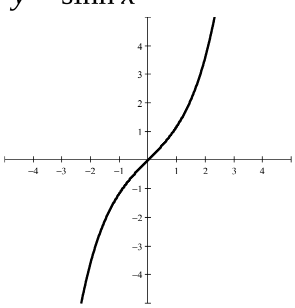
[e] graph of $y = \sinh x$ goes left and downward in quadrant 3

graph of $y = \sinh x$ goes right and upward in quadrant 1 ★

graph of $y = \cosh x$ goes left and upward in quadrant 2

graph of $y = \cosh x$ goes right and upward in quadrant 1

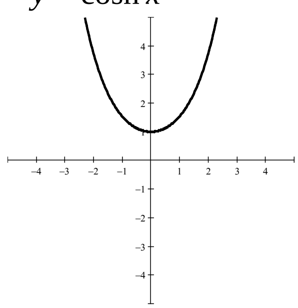
[f] $y = \sinh x$



Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

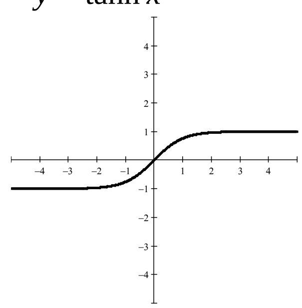
$y = \cosh x$



Domain: $(-\infty, \infty)$

Range: $[1, \infty)$

$y = \tanh x$



Domain: $(-\infty, \infty)$

Range: $(-1, 1)$

Horizontal Asymptotes:

$y = \pm 1$

[g] $y = \sinh x$ and $y = \tanh x$ are one-to-one since they satisfy the horizontal line test (for each value y, there is at most one corresponding value of x).

[h] $y = \cosh x$ can be made one-to-one by restricting its domain to only $(-\infty, 0]$ or $[0, \infty)$.

Following the trend towards positive numbers in other cases (eg. $y = \cos x$ and $y = x^2$), $[0, \infty)$ is preferred.

[6]

$\sinh x = 1 \quad \Rightarrow \quad x = \ln(1 + \sqrt{2})$

$\cosh x = 1 \quad \Rightarrow \quad x = 0$ ★

[7]

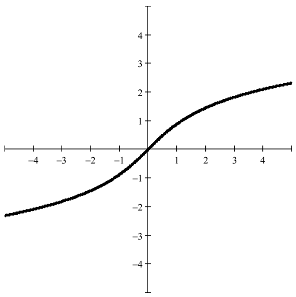
[a] $y = \cosh^{-1} x$ if and only if $x = \cosh y$ and $y \geq 0$

$y = \tanh^{-1} x$ if and only if $x = \tanh y$

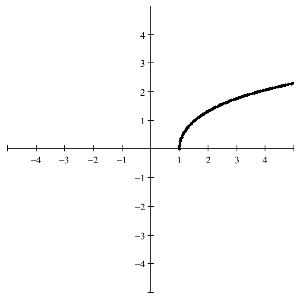
[b] ★

	$y = \sinh^{-1} x$	$y = \cosh^{-1} x$	$y = \tanh^{-1} x$
Domain	$(-\infty, \infty)$	$[1, \infty)$	$(-1, 1)$
Range	$(-\infty, \infty)$	$[0, \infty)$	$(-\infty, \infty)$
$x-$intercept(s)	$(0, 0)$	$(1, 0)$	$(0, 0)$
$y-$intercept	$(0, 0)$	none	$(0, 0)$
			Vertical Asymptotes: $x = \pm 1$

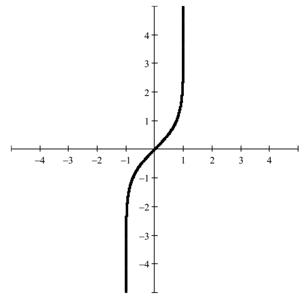
[c] $y = \sinh^{-1} x$



$y = \cosh^{-1} x$



$y = \tanh^{-1} x$



[d] [i] $f(g(x)) = x$ ★

[ii] $g(f(x)) = x$ ★

[e] $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$

SELECTED SOLUTIONS

[0]

[c] $\sinh x = \dfrac{e^x - e^{-x}}{2} = 0$

$e^x - e^{-x} = 0$

$e^{-x}(e^{2x} - 1) = 0$

Since $e^{-x} > 0$ for all real x,

$e^{2x} - 1 = 0$

$e^{2x} = 1$

$2x = \ln 1 = 0$

$x = 0$

[d] If $x > 0$,

$e^x > 1 > e^{-x}$,

$e^x - e^{-x} > 0$,

$\sinh x = \dfrac{e^x - e^{-x}}{2} > 0$

[e] As $x \to \infty$,

$e^x \to \infty$ and $e^{-x} \to 0$,

$\sinh x = \dfrac{e^x - e^{-x}}{2} \to \dfrac{\infty - 0}{2} \to \dfrac{\infty}{2} \to \infty$

[1]

[b] $\cosh(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^x}{2} = \dfrac{e^x + e^{-x}}{2} = \cosh x$

$\boxed{\text{Similarly, } \cos(-x) = \cos x}$

[e] $2\sinh x \cosh x = 2\left(\dfrac{e^x + e^{-x}}{2}\right)\left(\dfrac{e^x - e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

$\boxed{\text{Similarly, } \sin 2x = 2\sin x \cos x}$

[j] $\sinh(\ln x) = \dfrac{e^{\ln x} - e^{-\ln x}}{2} = \dfrac{e^{\ln x} - e^{\ln x^{-1}}}{2} = \dfrac{x - x^{-1}}{2} \cdot \dfrac{x}{x} = \dfrac{x^2 - 1}{2x}$

[2]

[a] $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Trying the hyperbolic counterpart,

$$\sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4} = \frac{2e^{x+y} - 2e^{-x-y}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$$= \sinh(x+y)$$

[3]

[b] [i] $\tanh 0 = \dfrac{\sinh 0}{\cosh 0} = \dfrac{0}{1} = 0$

Since the domains of $\sinh x$ and $\cosh x$ are both $(-\infty, \infty)$ (see **[0][b]**),

and the denominator $\cosh x$ can never be 0 (see **[0][c]**),

$\tanh x$ has domain $(-\infty, \infty)$.

$\tanh x = \dfrac{\sinh x}{\cosh x} = 0 \Rightarrow \sinh x = 0 \Rightarrow x = 0$ (see **[0][c]**)

[ii] $\tanh 0 = \dfrac{e^0 - e^0}{e^0 + e^0} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0$

Since e^x and e^{-x} are defined for all real x,

and $e^x > 0$ and $e^{-x} > 0$ for all real x,

therefore, the denominator $e^x + e^{-x} > 0$ for all real x,

so, $\tanh x$ is defined for all real x, and has domain $(-\infty, \infty)$.

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 0 \Rightarrow e^x - e^{-x} = 0 \Rightarrow x = 0$ (see **[0][c]**)

[c] [i] $\tanh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = \dfrac{e^{-x} - e^x}{e^{-x} + e^x} = \dfrac{-(e^x - e^{-x})}{e^x + e^{-x}} = -\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = -\tanh x$

$\boxed{\text{Similarly, } \tan(-x) = -\tan x}$

[ii] $\tanh(-x) = \dfrac{\sinh(-x)}{\cosh(-x)} = \dfrac{-\sinh x}{\cosh x} = -\dfrac{\sinh x}{\cosh x} = -\tanh x$

[f] [ii] $\operatorname{sech}(2\ln 3) = \dfrac{2}{e^{2\ln 3} + e^{-2\ln 3}} = \dfrac{2}{e^{\ln 3^2} + e^{\ln 3^{-2}}} = \dfrac{2}{3^2 + 3^{-2}} \cdot \dfrac{3^2}{3^2} = \dfrac{18}{82} = \dfrac{9}{41}$

[4]

[b] [i] $\cosh 2x = \cosh^2 x + \sinh^2 x = \cosh^2 x + \cosh^2 x - 1 = 2\cosh^2 x - 1$

$\boxed{\text{Similarly, } \cos 2x = 2\cos^2 x - 1}$

[c] [i] $\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \Rightarrow \coth^2 x - 1 = \operatorname{csch}^2 x$

$\boxed{\text{In contrast, } \cot^2 x + 1 = \csc^2 x}$

$\boxed{\text{There is a switch in addition versus subtraction here comparing trigonometric versus hyperbolic}}$

[d] $\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{1}{4} = \frac{5}{4} \Rightarrow \cosh x = \pm\frac{\sqrt{5}}{2}$

Since $\cosh x > 0$ for all real x (see **[0][d]**), $\cosh x = \frac{\sqrt{5}}{2}$.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}}{\frac{\sqrt{5}}{2}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

[5]

[b] $x > 0 \Rightarrow y = \sinh x > 0 \Rightarrow$ part of graph of $y = \sinh x$ is in quadrant 1

[c] $x > 0 \Rightarrow \sinh x > 0, \cosh x > 0 \Rightarrow \tanh x = \frac{\sinh x}{\cosh x} > 0 \Rightarrow$ part of graph of $y = \tanh x$ is in quadrant 1

[d] $\cosh(-x) = \cosh x \Rightarrow y = \cosh x$ is even and symmetric over the $y-$axis

[e] As $x \to \infty$, $y = \sinh x \to \infty$ so the graph of $y = \sinh x$ goes right and upward in quadrant 1

[6]

$$\cosh x = 1 \Rightarrow \frac{e^x + e^{-x}}{2} = 1 \Rightarrow \frac{e^x + \frac{1}{e^x}}{2} = 1 \Rightarrow \frac{z + \frac{1}{z}}{2} = 1 \Rightarrow z^2 + 1 = 2z \Rightarrow z^2 - 2z + 1 = 0$$

$$\Rightarrow (z-1)^2 = 0 \Rightarrow z = 1 \Rightarrow x = \ln z = 0$$

[7]

[b] The inverse of a function is the result of
swapping the inputs ($x-$values) of the function with its outputs ($y-$values).

[c] The graph of the inverse of a one-to-one function is
the reflection of the graph of the function over the line $y = x$.

[d] [i] $f(g(x)) = \dfrac{e^{\ln(x+\sqrt{x^2+1})} - e^{-\ln(x+\sqrt{x^2+1})}}{2} = \dfrac{x+\sqrt{x^2+1} - \dfrac{1}{x+\sqrt{x^2+1}}}{2} = \dfrac{(x+\sqrt{x^2+1})^2 - 1}{2(x+\sqrt{x^2+1})}$

$= \dfrac{x^2 + 2x\sqrt{x^2+1} + x^2 + 1 - 1}{2(x+\sqrt{x^2+1})} = \dfrac{2x^2 + 2x\sqrt{x^2+1}}{2(x+\sqrt{x^2+1})} = \dfrac{2x(x+\sqrt{x^2+1})}{2(x+\sqrt{x^2+1})} = x$

[ii] $g(f(x)) = \ln(\sinh x + \sqrt{\sinh^2 x + 1}) = \ln(\sinh x + \sqrt{\cosh^2 x}) = \ln(\sinh x + \cosh x)$

$= \ln e^x = x$

[iii] $x = \sinh y \Rightarrow x = \dfrac{e^y - e^{-y}}{2} \Rightarrow x = \dfrac{e^y - \dfrac{1}{e^y}}{2} \Rightarrow x = \dfrac{z - \dfrac{1}{z}}{2} \Rightarrow 2xz = z^2 - 1$

$\Rightarrow z^2 - 2xz - 1 = 0 \Rightarrow z = \dfrac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} \Rightarrow z = \dfrac{2x \pm \sqrt{4x^2+4}}{2}$

$\Rightarrow z = \dfrac{2x \pm 2\sqrt{x^2+1}}{2} \Rightarrow z = x \pm \sqrt{x^2+1}$

Since $x^2 + 1 > x^2 \geq 0$,

if $x \geq 0$, then $0 < x < \sqrt{x^2+1}$, so $x - \sqrt{x^2+1} < 0$

whereas $x + \sqrt{x^2+1} > 0$,

and if $x < 0$, then $0 < |x| = -x < \sqrt{x^2+1}$, so $x - \sqrt{x^2+1} < 0 - \sqrt{x^2+1} < 0$

whereas $x + \sqrt{x^2+1} > x + (-x) = 0$.

Since $z = e^y > 0$, $z = x + \sqrt{x^2+1} \Rightarrow y = \ln z = \ln(x+\sqrt{x^2+1})$