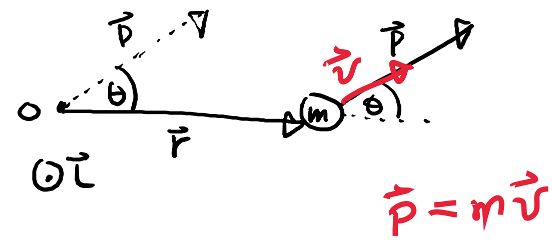
Angular Momentum Notes

Angular Momentum

Def: If a particle has linear momentum $\vec{p}$ at position $\vec{r}$ relative to origin "o", then its angular momentum $\vec{L}$ relative to origin "o" is given by:

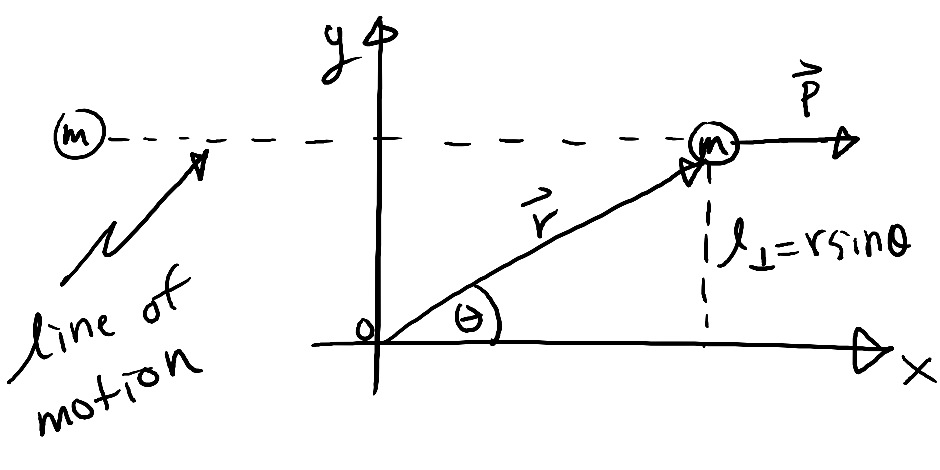
$\boxed{\vec{L} = \vec{r} \times \vec{p}}$ Angular Mometum



$\boxed{L = rp\sin\theta}$ Magnitude of $\vec{L}$

a) The magnitude of and direction of $\vec{L}$ depend on the choice of origin.
b) The direction of $\vec{L}$ is given by the RHR.
c) The SI unit of $\vec{L}$ is the m.kg. m/s = kg m^2/s

Ex. A particle of mass 'm' is moving along the x-y plane with constant velocity $\vec{v}$ parallel to the x-axis at at y = ℓ. Find $\vec{L}$ relative to the origin.



$\vec{L}_o = \vec{r} \times \vec{p}$

$L_o = rp\sin\theta$

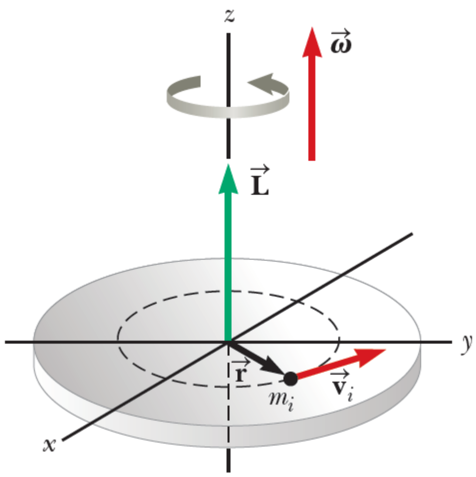
$L_o = p(r\sin\theta)$

$\boxed{L_o = p\,\ell_\perp}$

$\boxed{\vec{L}_o = (mv\ell_\perp)\hat{k}}$

Angular Momentum for a Rotating Body

Consider a body rotating about the z-axis with constant angular velocity $\vec{\omega}$.



Let's look at the angular momentum of an m_i particle at a distance r_i from the axis of rotation:

$$\vec{L_i} = \vec{r_i} \times \vec{p_i} = r_i m_i v_i \hat{k}$$

$$L_{iz} = r_i m_i r_i w_i$$

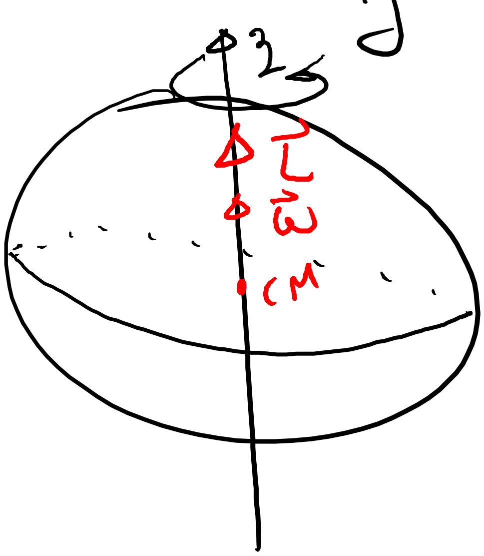
$$L_{iz} = m_i r_i^2 w$$

$$L_z = \sum L_{iz} = \left(\sum m_i r_i^2\right) w$$

$\boxed{L_z = I w_z}$ Angular Momentum about the z-axis

$\boxed{\vec{L}_{sys} = I\vec{\omega}}$ Angular Momentum vector

Ex. Rotating Sphere



$$\vec{L} = I\vec{\omega}$$

$$\boxed{\vec{L} = \left(\frac{2}{5}MR^2\right)\omega\hat{k}}$$

<u>Net Torque on a System</u>

Consider a system rotating about an axis due to a net external torque:

$$\sum \vec{\tau}_{ext} = I\vec{\alpha}$$

This net torque will cause the system to experience an angular acceleration:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Which means that $\vec{\omega}$ change. If $\overrightarrow{\omega}$ changes, then so will $\vec{L}_{sys}$ also change since:

$$\vec{L}_{sys} = I\vec{\omega}$$

Question, is how will $\vec{L}_{sys}$ of the system change? Let's differentiate $\vec{L}_{sys} = I\vec{\omega}$ to see how $\vec{L}_{sys}$ changes:

$$\frac{d\vec{L}_{sys}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

However,

$$\sum \vec{\tau}_{ext} = I\vec{\alpha}$$

Therefore,

(1) $\boxed{\sum \vec{\tau}_{ext} = \frac{d\vec{L}_{sys}}{dt}}$ Net Torque on a System

This equation is the rotational analog of $\sum \vec{F}_{ext} = \frac{d\vec{P}_{sys}}{dt}$

If $\sum \vec{\tau}_{ext} = 0 = \frac{d\vec{L}_{sys}}{dt}$, then

$$\boxed{\begin{gathered}\vec{L}_{sys} = constant \\ \overrightarrow{\Delta L}_{sys} = 0 \\ \vec{L}_i = \vec{L}_f\end{gathered}}$$ Conservation of Angular Momentum

Thus,

$$\boxed{\begin{gathered}E_i = E_f \\ \vec{p}_i = \vec{p}_f \\ \vec{L}_i = \vec{L}_f\end{gathered}}$$ Conservation Laws for an Isolated System

In component form:

$$\vec{L}_i = \vec{L}_f \Longrightarrow \begin{cases} L_{ix} = L_{fx} \ \ if \sum \tau_{ext(x)} = 0 \\ L_{iy} = L_{fy} \ \ if \sum \tau_{ext(y)} = 0 \\ L_{iz} = L_{fz} \ \ if \sum \tau_{ext(z)} = 0 \end{cases}$$

Going back to Equation (1),

$$\sum \vec{\tau}_{ext}\,(t) = \frac{d\vec{L}_{sys}}{dt}$$

$$\sum \vec{\tau}_{ext}(t) = \frac{d\vec{L}_{sys}}{dt}$$
$$d\vec{L}_{sys} = \sum \vec{\tau}_{ext}(t)\,dt$$
$$\int_{\vec{L}_i}^{\vec{L}_f} d\vec{L}_{sys} = \int_{t_i}^{t_f} \sum \vec{\tau}_{ext}\,(t)dt$$
$$\Delta\vec{L}_{sys} = \int_{t_i}^{t_f} \sum \vec{\tau}_{ext}\,(t)dt$$

In a collision we define $\Delta\vec{L}_{sys}$ to be the angular impulse:

$\boxed{\vec{J} = \Delta\vec{L}_{sys}}$ Angular Impulse Vector

Thus,

$\boxed{\vec{J} = \Delta\vec{L}_{sys} = \int_{t_i}^{t_f} \sum \vec{\tau}_{ext}\,(t)dt}$ Impulse Angular- MomentumTheorem

For an isolated system:

$\sum \vec{\tau}_{ext}\,(t)$=0 $\Longrightarrow \vec{J} = \Delta\vec{L}_{sys}$=0 (Conservation of $\vec{L}_{sys}$)

However,

1. If $\sum \vec{\tau}_{ext}(t)$ is negligibly small, then $\vec{J} = \Delta\vec{L}_{sys} \approx 0$
2. If $\sum \vec{\tau}_{ext}(t)$ acts for a very short period of time, then $\vec{J} = \Delta\vec{L}_{sys} \approx 0$