Applied Finite Mathematics

Third Edition, 2016

Revised and Updated by

Roberta Bloom

De Anza College

Based on Second Edition 1996

By Rupinder Sekhon

De Anza College

Preface to Third Edition

This third edition of Applied Finite Mathematics, 2016, updates the second edition of Applied Finite Mathematics by Rupinder Sekhon, 1996. Rupinder Sekhon is an Instructor Emeritus in the Mathematics Department at De Anza College in Cupertino, CA.

This textbook is intended to provide learning materials that primarily match required topics in the course outline for Finite Mathematics at De Anza College, as of 2016. Due to this focus, it does not contain some topics included in Finite Math textbooks intended for more widespread use across a broader spectrum of colleges.

Some of the updates and changes for the third edition include

- Replacing or updating some homework problems or examples with current data or contexts
- Additional homework problems in some chapters to expand the types of problems or contexts of problems available, or to provide additional practice.
- To provide students with expanded insight into how mathematical models are used in the age of analytics and big data, new material has been added to some chapters or sections. In particular, new sections examine several large scale applications of linear programming (and related integer programming), and applications of Markov Chains; in addition the section about cryptography has an expanded introduction.
- Modification of existing content to enhance explanations in some sections.
- A new chapter covers exponential and logarithmic functions. As of 2016, these topics are required for courses that need to conform with the course descriptor (C-ID) for Finite Mathematics in California. Several sections in this chapter are adapted from the sources below, under Creative Commons Attribution licenses:

  David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “Chapter 4: Exponential and Logarithmic Functions,” Creative Commons CC BY-SA 3.0 license.

  Precalculus, from OpenStax, which can be downloaded for free at https://openstax.org/details/precalculus , Creative Commons Attribution License v4.0
- A new chapter with instructions for TI-83+ and TI-84+ graphing calculators, including use of the TVM (time value of money) solver, replaces material for the now discontinued TI-85 and TI-86 calculator.
- Many of the original equations and diagrams are included unchanged, but some have had to be redone. Updating a mathematics textbook with equations and diagrams, presented some unanticipated problems with technology compatibility, due to the time and software changes between today and creation of the original material. Although some earlier errors have been corrected, some new ones were almost certainly introduced; the author appreciates readers notifying her at **bloomroberta@fhda.edu** about corrections that are needed.

I would like to thank my family for their support and their contributions, based on work in their fields, to my research on modern applications of the methods included in this book. Thank you to Doli Bambhania and Barbara Illowsky for their practical advice, and to De Anza College and Jerry Rosenberg, Dean of the Physical Science, Mathematics, and Engineering Division for support and resources. Most importantly, I’d like to thank Rupinder Sekhon for his work as the original author and for putting this book under a Creative Commons Attribution License so that it could be expanded and can continue to be available for students to use free online or at low cost in print. I also acknowledge and extend my appreciation to several authors whose material, under Creative Commons Attribution Licenses, has been adapted for some portions of this book; credit for such material, in accordance with the licenses, has been included in this book at the places where such material, or adaptation thereof, appears.

Roberta Bloom
Mathematics Department, De Anza College
Cupertino, CA
2016

Table of Contents

The problem set for each chapter follows the end of the section.
The review sections consist of additional practice problems for each section.

CHAPTER 1

Chapter 1: Linear Equations

In this chapter, you will learn to:

1. *Graph a linear equation.*
2. *Find the slope of a line.*
3. *Determine an equation of a line.*
4. *Solve linear systems.*
5. *Do application problems using linear equations.*

1.1 Graphing a Linear Equation

In this section, you will learn to:

1. *Graph a line when you know its equation*
2. *Graph a line when you are given its equation in parametric form*
3. *Graph and find equations of vertical and horizontal lines*

GRAPHING A LINE FROM ITS EQUATION

Equations whose graphs are straight lines are called **linear equations.** The following are some examples of linear equations:

$2x - 3y = 6$, $3x = 4y - 7$, $y = 2x - 5$, $2y = 3$, and $x - 2 = 0$.

A line is completely determined by two points. Therefore, to graph a linear equation we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

◆***Example 1*** Graph the line: $y = 3x + 2$

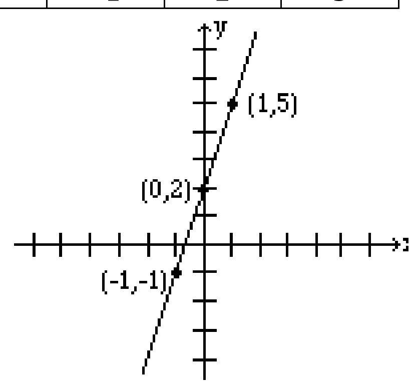
Solution: We need to find the coordinates of at least two points. We arbitrarily choose $x = -1$, $x = 0$, and $x = 1$.

If $x = -1$, then $y = 3(-1) + 2$ or -1. Therefore, $(-1, -1)$ is a point on this line.

If $x = 0$, then $y = 3(0) + 2$ or $y = 2$. Hence the point $(0, 2)$.

If $x = 1$, then $y = 5$, and we get the point $(1, 5)$. Below, the results are summarized, and the line is graphed.

x	–1	0	1
y	–1	2	5



◆***Example 2*** Graph the line: 2x + y = 4

Solution: Again, we need to find coordinates of at least two points.

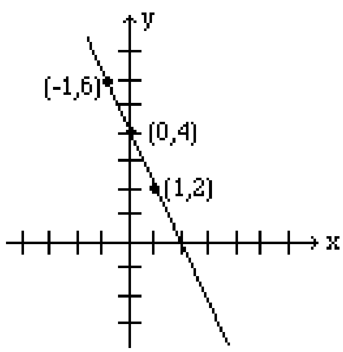
We arbitrarily choose x = –1, x = 0, and y = 2.

If x = –1, then 2(–1) + y = 4 which results in y = 6.
Therefore, (–1, 6) is a point on this line.

If x = 0, then 2(0) + y = 4, which results in y = 4. Hence the point (0, 4).

If y = 2, then 2x + 2 = 4, which yields x = 1, and gives the point (1, 2).
The table below shows the points, and the line is graphed.

x	–1	0	1
y	6	4	2



INTERCEPTS

The points at which a line crosses the coordinate axes are called the **intercepts**.
When graphing a line by plotting two points, using the intercepts is often preferred because they are easy to find.

- To find the value of the x-intercept, we let y = 0
- To find the value of the y-intercept, we let x = 0.

◆***Example 3*** Find the intercepts of the line: 2x – 3y = 6, and graph.

Solution: To find the x-intercept, let y = 0 in the equation, and solve for x.

2x – 3(0) = 6

2x – 0 = 6

2x = 6

x = 3 Therefore, the x-intercept is the point (3,0).

To find the y-intercept, let x = 0 in the equation, and solve for y.

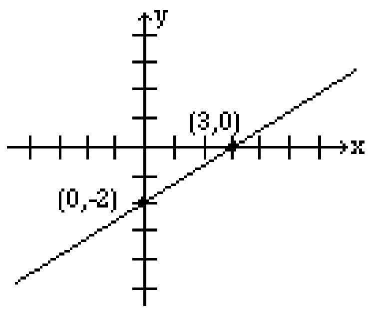
$2(0) - 3y = 6$

$0 - 3y = 6$

$-3y = 6$

$y = -2$ Therefore, the y-intercept is the point (0, –2).

To graph the line, plot the points for the x-intercept (3,0) and the y-intercept (0, –2), and use them to draw the line.



GRAPHING A LINE FROM ITS EQUATION IN PARAMETRIC FORM

In higher math, equations of lines are sometimes written in parametric form. For example, $x = 3 + 2t$, $y = 1 + t$. The letter t is called the **parameter**, or the dummy variable.

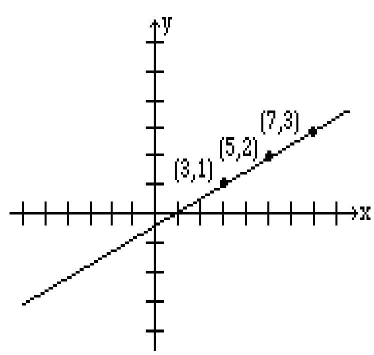
Parametric lines can be graphed by finding values for x and y by substituting numerical values for t. Plot the points using their (x,y) coordinates and use the points to draw the line.

◆***Example 4*** Graph the line given by the parametric equations: $x = 3 + 2t$, $y = 1 + t$

Solution: Let t = 0, 1 and 2; for each value of t find the corresponding values for x and y.

The results are given in the table below.

t	x	y	Point on Line
0	3	1	(3,1)
1	5	2	(5,2)
2	7	3	(7,3)



HORIZONTAL AND VERTICAL LINES

When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

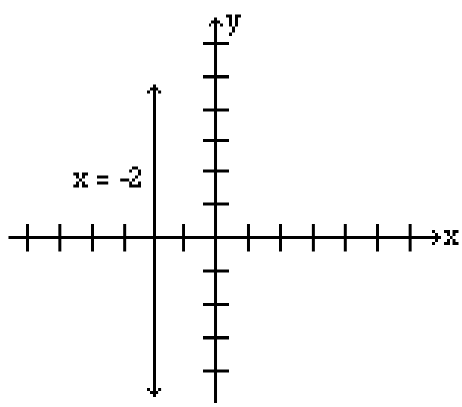
The graph of the line $x = a$, where a is a constant, is a vertical line that passes through the point (a, 0). Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.

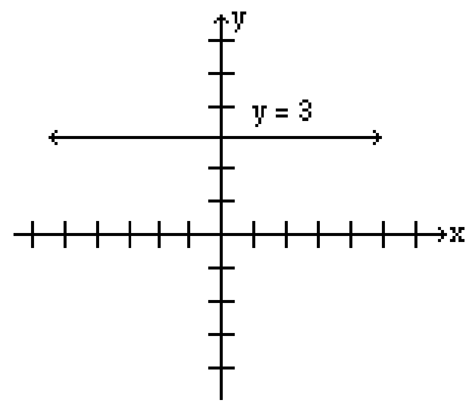
The graph of the line $y = b$, where b is a constant, is a horizontal line that passes through the point (0, b). Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

◆***Example 5*** Graph the lines: $x = –2$, and $y = 3$.

Solution: The graph of the line $x = –2$ is a vertical line that has the x-coordinate –2 no matter what the y-coordinate is. The graph is a vertical line passing through point (–2, 0).

The graph of the line $y = 3$, is a horizontal line that has the y-coordinate 3 regardless of what the x-coordinate is. Therefore, the graph is a horizontal line that passes through point (0, 3).





Note: Most students feel that the coordinates of points must always be integers. This is not true, and in real life situations, not always possible. Do not be intimidated if your points include numbers that are fractions or decimals.

SECTION 1.1 PROBLEM SET: GRAPHING A LINEAR EQUATION

Work the following problems.

1) Is the point (2, 3) on the line $5x - 2y = 4$?	2) Is the point (1, – 2) on the line $6x - y = 4$?
3) For the line $3x - y = 12$, complete the following ordered pairs. (2,) (, 6) (0,) (, 0)	4) For the line $4x + 3y = 24$, complete the following ordered pairs. (3,) (, 4) (0,) (, 0)
5) Graph $y = 2x + 3$	6) Graph $y = -3x + 5$
7) Graph $y = 4x - 3$	8) Graph $x - 2y = 8$
9) Graph $2x + y = 4$	10) Graph $2x - 3y = 6$

SECTION 1.1 PROBLEM SET: GRAPHING A LINEAR EQUATION

11) Graph $2x + 4 = 0$	12) Graph $2y - 6 = 0$
13) Graph the following three equations on the same set of coordinate axes. $y = x + 1$ $y = 2x + 1$ $y = -x + 1$	14) Graph the following three equations on the same set of coordinate axes. $y = 2x + 1$ $y = 2x$ $y = 2x - 1$
15) Graph the line using the parametric equations $x = 1 + 2t$, $y = 3 + t$	16) Graph the line using the parametric equations $x = 2 - 3t$, $y = 1 + 2t$

1.2 Slope of a Line

In this section, you will learn to:

1. *Find the slope of a line.*
2. *Graph the line if a point and the slope are given.*

In the last section, we learned to graph a line by choosing two points on the line. A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the **slope** of the line.

From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\frac{rise}{run}, \quad \frac{vertical\ \ change}{horizontal\ \ change}, \quad \frac{\Delta y}{\Delta x} \text{ etc.}$$

We give a precise definition.

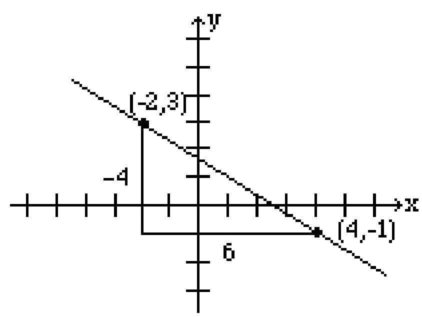
Definition: If (x_1, y_1) and (x_2, y_2) are two different points on a line, the **slope** of the line is

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

◆***Example 1*** Find the slope of the line passing through points (–2, 3) and (4, –1), and graph the line.

Solution: Let $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, -1)$, then the slope is

$$m = \frac{-1-3}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3}$$



To give the reader a better understanding, both the vertical change, –4, and the horizontal change, 6, are shown in the above figure.

When two points are given, it does not matter which point is denoted as (x_1, y_1) and which (x_2, y_2). The value for the slope will be the same.

In Example 1, if we instead choose $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-2, 3)$, then we will get the same value for the slope as we obtained earlier.
The steps involved are as follows.

$$m = \frac{3-(-1)}{-2-4} = \frac{4}{-6} = -\frac{2}{3}$$

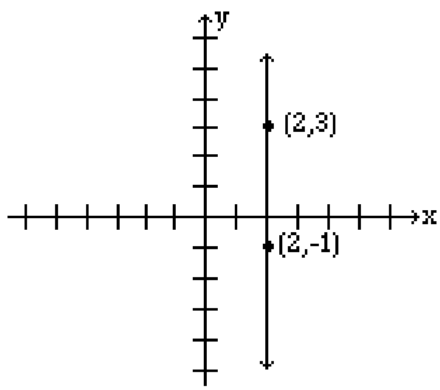
The student should further observe that

- if a line rises when going from left to right, then it has a positive slope. In this situation, as the value of x increases, the value of y also increases
- if a line falls going from left to right, it has a negative slope; as the value of x increases, the value of y decreases.

◆***Example 2*** Find the slope of the line that passes through the points (2, 3) and (2, –1), and graph.

Solution: Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (2, -1)$ then the slope is

$$m = \frac{-1-3}{2-2} = -\frac{4}{0} = \text{undefined.}$$

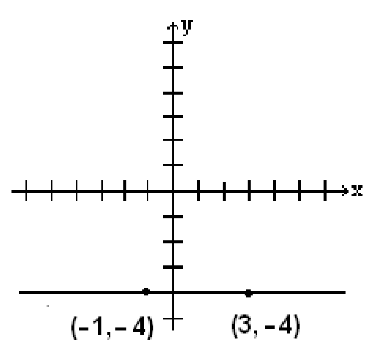


Note: The slope of a vertical line is undefined.

◆***Example 3*** Find the slope of the line that passes through the points (–1, –4) and (3, –4)

Solution: Let $(x_1, y_1) = (-1, -4)$ and $(x_2, y_2) = (3, -4)$, then the slope is

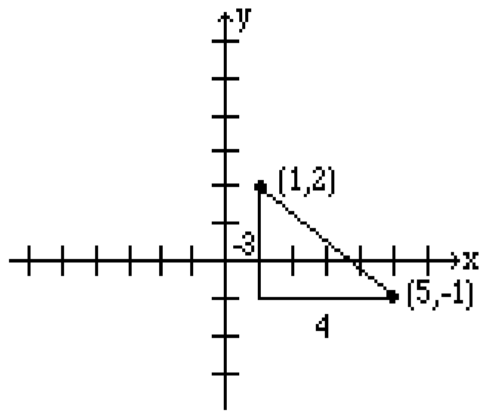
$$m = \frac{-4-(-4)}{3-(-1)} = \frac{0}{4} = 0$$



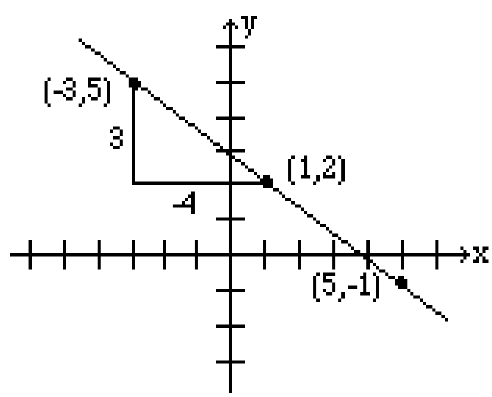
Note: The slope of a horizontal line is 0

◆***Example 4*** Graph the line that passes through the point (1, 2) and has slope $-\frac{3}{4}$.

Solution: Slope equals $\frac{\text{rise}}{\text{run}}$. The fact that the slope is $\frac{-3}{4}$, means that for every rise of –3 units (fall of 3 units) there is a run of 4. So if from the given point (1, 2) we go down 3 units and go right 4 units, we reach the point (5, –1). The graph is obtained by connecting these two points.



Alternatively, since $\frac{3}{-4}$ represents the same number, the line can be drawn by starting at the point (1,2) and choosing a rise of 3 units followed by a run of –4 units. So from the point (1, 2), we go up 3 units, and to the left 4, thus reaching the point (–3, 5) which is also on the same line. See figure below.



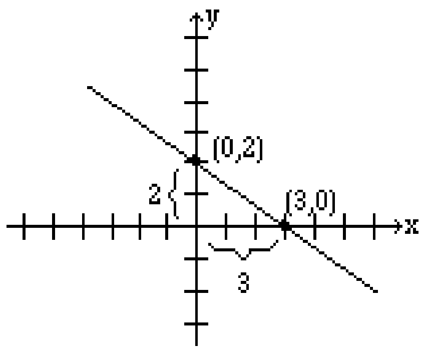
◆***Example 5*** Find the slope of the line 2x + 3y = 6.

Solution: In order to find the slope of this line, we will choose any two points on this line.

Again, the selection of x and y intercepts seems to be a good choice. The x-intercept is (3, 0), and the y-intercept is (0, 2). Therefore, the slope is

$$m = \frac{2-0}{0-3} = -\frac{2}{3}.$$

The graph below shows the line and the x-intercepts and y-intercepts:



◆***Example 6*** Find the slope of the line y = 3x + 2.

Solution: We again find two points on the line. Say (0, 2) and (1, 5).

Therefore, the slope is $m = \frac{5-2}{1-0} = \frac{3}{1} = 3$.

Look at the slopes and the y-intercepts of the following lines.

The line	slope	y-intercept
y = 3x + 2	3	2
y = – 2x + 5	– 2	5
y = (3/2) x – 4	3/2	– 4

It is no coincidence that when an equation of the line is solved for y, the coefficient of the x term represents the slope, and the constant term represents the y-intercept.

In other words, for the line y = mx + b, m is the slope, and b is the y-intercept.

◆***Example 7*** Determine the slope and y-intercept of the line 2x + 3y = 6.

Solution: We solve for y:

2x + 3y = 6

3y = –2x + 6

y = (–2/3) x + 2

The slope = the coefficient of the x term = – 2/3.
The y-intercept = the constant term = 2.

SECTION 1.2 PROBLEM SET: SLOPE OF A LINE

Find the slope of the line passing through the following pair of points.

1) (2, 3) and (5, 9)	2) (4, 1) and (2, 5)
3) (– 1, 1) and (1, 3)	4) (4, 3) and (– 1, 3)
5) (6, – 5) and (4, – 1)	6) (5, 3) and (– 1, – 4)
7) (3, 4) and (3, 7)	8) (– 2, 4) and (– 3, – 2)
9) (– 3, – 5) and (– 1, – 7)	10) (0, 4) and (3, 0)

SECTION 1.2 PROBLEM SET: SLOPE OF A LINE

Determine the slope of the line from the given equation of the line.

11) $y = -2x + 1$	12) $y = 3x - 2$
13) $2x - y = 6$	14) $x + 3y = 6$
15) $3x - 4y = 12$	16) What is the slope of the x-axis? What is the slope of the y-axis?

Graph the line that passes through the given point and has the given slope.

17) (1, 2) and $m = -3/4$	18) (2, – 1) and $m = 2/3$
19) (0, 2) and $m = -2$	20) (2, 3) and $m = 0$

1.3 Determining the Equation of a Line

In this section, you will learn to:

1. *Find an equation of a line if a point and the slope are given.*
2. *Find an equation of a line if two points are given.*

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points, or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms, the **slope-intercept form**, the **point-slope form,** or the **standard form**. We will discuss each of them in this section.

A line is completely determined by two points, or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

THE SLOPE-INTERCEPT FORM OF A LINE: **y = mx + b**

In the last section we learned that the equation of a line whose slope = m and y-intercept = b is **y = mx + b**. This is called the **slope-intercept form** of the line and is the most commonly used form.

◆***Example 1*** Find an equation of a line whose slope is 5, and y-intercept is 3.

Solution: Since the slope is m = 5, and the y- intercept is b = 3, the equation is y = 5x + 3.

◆***Example 2*** Find the equation of the line that passes through the point (2, 7) and has slope 3.

Solution: Since m = 3, the partial equation is y = 3x + b.

Now b can be determined by substituting the point (2, 7) in the equation y = 3x + b.

7 = 3(2) + b

b = 1

Therefore, the equation is y = 3x + 1.

◆***Example 3*** Find an equation of the line that passes through the points (–1, 2), and (1, 8).

Solution: $m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$. So the partial equation is y = 3x + b

We can use either of the two points (–1, 2) or (1, 8), to find b. Substituting (–1, 2) gives

2 = 3(–1) + b

5 = b

So the equation is y = 3x +5.

◆***Example 4*** Find an equation of the line that has x-intercept 3, and y-intercept 4.

Solution: x-intercept = 3, and y-intercept = 4 correspond to the points (3, 0), and (0, 4), respectively.

$$m = \frac{4-0}{0-3} = -\frac{4}{3}$$

We are told the y intercept is 4; thus b = 4

Therefore, the equation is $y = -4/3\, x + 4$.

THE POINT-SLOPE FORM OF A LINE: $y - y_1 = m(x - x_1)$

The **point-slope** form is useful when we know two points on the line and want to find the equation of the line.

Let L be a line with slope m, and known to contain a specific point (x_1, y_1). If (x, y) is any other point on the line L, then the definition of a slope leads us to the **point-slope form** or point-slope formula.

The slope is $\dfrac{y - y_1}{x - x_1} = m$

Multiplying both sides by $(x - x_1)$ gives the point-slope form:

$$\mathbf{y - y_1 = m(x - x_1)}$$

◆***Example 5*** Find the point-slope form of the equation of a line that has slope 1.5 and passes through the point (12,4).

Solution: Substituting the point $(x_1,y_1) = (12,4)$ and m= 1.5 in the point-slope formula, we get

$$\mathbf{y - y_1 = m(x - x_1)}$$
$$y - 4 = 1.5(x - 12)$$

The student may be tempted to simplify this into the slope intercept form y = mx + b. But since the problem specifically requests **point-slope** form we will not simplify it.

THE STANDARD FORM OF A LINE: Ax + By = C

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y_1 = m(x - x_1)$, or if we know the equation of the line in slope-intercept form y = mx + b, we can simplify the formula to have all terms for the x and y variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the **standard form** of the line: **Ax + By = C**.

◆***Example 6*** Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope –3/5.

Solution: Substituting the point (2, 3) and m= – 3/5 in the point-slope formula, we get

$$y - 3 = -3/5(x - 2)$$

Multiplying both sides by 5 gives us

$$5(y - 3) = -3(x - 2)$$

$$5y - 15 = -3x + 6$$

$$3x + 5y = 21 \quad \text{Standard Form}$$

◆***Example 7*** Find the standard form of the line that passes through the points (1, –2), and (4, 0).

Solution: First we find the slope: $m = \dfrac{0-(-2)}{4-1} = \dfrac{2}{3}$

Then, the point-slope form is: $y - (-2) = 2/3(x - 1)$

Multiplying both sides by 3 gives us

$$3(y + 2) = 2(x - 1)$$

$$3y + 6 = 2x - 2$$

$$-2x + 3y = -8$$

$$2x - 3y = 8 \quad \text{Standard Form}$$

We should always be able to convert from one form of an equation to another. For example, if we are given a line in the slope-intercept form, we should be able to express it in the standard form, and vice versa.

◆***Example 8*** Write the equation $y = -2/3\ x + 3$ in the standard form.

Solution: Multiplying both sides of the equation by 3, we get

$$3y = -2x + 9$$

$$2x + 3y = 9 \quad \text{Standard Form}$$

◆***Example 9*** Write the equation $3x - 4y = 10$ in the slope-intercept form.

Solution: Solving for y, we get

$$-4y = -3x + 10$$

$$y = 3/4\ x - 5/2 \quad \text{Slope Intercept Form}$$

Finally, we learn a very quick and easy way to write an equation of a line in the standard form. But first we must learn to find the slope of a line in the standard form by inspection.

By solving for y, it can easily be shown that the slope of the line $Ax + By = C$ is $-A/B$. The reader should verify this.

◆***Example 10*** Find the slope of the following lines, by inspection.

a) $3x - 5y = 10$ b) $2x + 7y = 20$ c) $4x - 3y = 8$

Solution: a) $A = 3$, $B = -5$, therefore, $m = -\frac{3}{-5} = \frac{3}{5}$

b) $A = 2$, $B = 7$, therefore, $m = -\frac{2}{7}$

c) $m = -\frac{4}{-3} = \frac{4}{3}$

Now that we know how to find the slope of a line in the standard form by inspection, our job in finding the equation of a line is going to be easy.

◆***Example 11*** Find an equation of the line that passes through (2, 3) and has slope – 4/5.

Solution: Since the slope of the line is – 4/5, we know that the left side of the equation is $4x + 5y$, and the partial equation is going to be

$$4x + 5y = c$$

Of course, c can easily be found by substituting for x and y.

$$4(2) + 5(3) = c$$

$$23 = c$$

The desired equation is

$$4x + 5y = 23.$$

If you use this method often enough, you can do these problems very quickly.

We summarize the forms for equations of a line below:

Slope Intercept form: $y = mx + b$,
where m = slope, b = y-intercept

Point Slope form: $y - y_1 = m(x - x_1)$,
where m = slope, (x_1,y_1) is a point on the line

Standard form: $Ax + By = C$

Horizontal Line: $y = b$
where b = y-intercept

Vertical Line: $x = a$
where a = x-intercept

SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

1) It passes through the point (3, 10) and has slope = 2.	2) It passes through point (4,5) and has m = 0.
3) It passes through (3, 5) and (2, – 1).	4) It has slope 3, and its y-intercept equals 2.
5) It passes through (5, – 2) and m = 2/5.	6) It passes through (– 5, – 3) and (10, 0).
7) It passes through (4, – 4) and (5, 3).	8) It passes through (7, – 2) ; its y-intercept is 5.
9) It passes through (2, – 5) and its x-intercept is 4.	10) Its a horizontal line through the point (2, – 1).

SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

11) It passes through (5, – 4) and (1, – 4).	12) It is a vertical line through the point (3, – 2).
13) It passes through (3, – 4) and (3, 4).	14) It has x-intercept = 3 and y-intercept = 4.

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

15) It passes through (3, – 1) and m = 2.	16) It passes through (– 2, 1) and m = – 3/2.
17) It passes through (– 4, – 2) and m = 3/4.	18) Its x-intercept equals 3, and m = – 5/3.

SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

19) It passes through (2, – 3) and (5, 1).	20) It passes through (1, – 3) and (– 5, 5).

Write an equation of the line satisfying the following conditions.
Write the equation in point slope form $y-y_1 = m\,(x-x_1)$

21) It passes through (2, – 3) and (5, 1).	22) It passes through (1, – 3) and (– 5, 2).
23) It passes through (6, –2) and (0, 2).	24) It passes through (8, 2) and (–7, –4).
25) It passes through (–12, 7) and has slope = –1/3.	26) It passes through (8, – 7) and has slope 3/4.

1.4 Applications

In this section, you will learn to use linear functions to model real-world applications

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations.

Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

◆***Example 1*** A taxi service charges $0.50 per mile plus a $5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?

Solution: x = distance traveled, in miles and y = cost in dollars

The cost of traveling 20 miles is y = (.50)(20) + 5 = 10 + 5 = 15

The cost of traveling x miles is y = (.50)(x) + 5 = .50x + 5

In this problem, $0.50 per mile is referred to as the **variable cost**, and the flat charge $5 as the **fixed cost**. Now if we look at our cost equation y = .50x + 5, we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.

◆***Example 2*** The variable cost to manufacture a product is $10 per item and the fixed cost $2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.

Solution: The variable cost of $10 per item tells us that m = 10.
The fixed cost represents the y-intercept. So b = 2500.

Therefore, the cost equation is y = 10x + 2500.

◆***Example 3*** It costs $750 to manufacture 25 items, and $1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.

Solution: We let x = the number of items manufactured, and let y = the cost.

Solving this problem is equivalent to finding an equation of a line that passes through the points (25, 750) and (50, 1000).

$$m = \frac{1000-750}{50-25} = 10$$

Therefore, the partial equation is y = 10x + b

By substituting one of the points in the equation, we get b = 500

Therefore, the cost equation is y = 10x + 500

To find the cost of 100 items, substitute x = 100 in the equation y = 10x + 500

So the cost = y = 10(100) + 500 = 1500

It costs $1500 to manufacture 100 items.

◆***Example 4*** The freezing temperature of water in Celsius is 0 degrees and in Fahrenheit 32 degrees. And the boiling temperatures of water in Celsius, and Fahrenheit are 100 degrees, and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.

Solution: Let us look at what is given.

Celsius	Fahrenheit
0	32
100	212

Again, solving this problem is equivalent to finding an equation of a line that passes through the points (0, 32) and (100, 212).

Since we are finding a linear relationship, we are looking for an equation y = mx + b, or in this case F = mC + b, where x or C represent the temperature in Celsius, and y or F the temperature in Fahrenheit.

$$\text{slope } m = \frac{212 - 32}{100 - 0} = \frac{9}{5}$$

The equation is $F = \frac{9}{5}C + b$

Substituting the point (0, 32), we get

$$F = \frac{9}{5}C + 32.$$

To convert 30 degrees Celsius into Fahrenheit, substitute C = 30 in the equation

$$F = \frac{9}{5}C + 32$$

$$F = \frac{9}{5}(30) + 32 = 86$$

◆***Example 5*** The population of Canada in the year 1980 was 24.5 million, and in the year 2010 it was 34 million. The population of Canada over that time period can be approximately modelled by a linear function. Let x represent time as the number of years after 1980 and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.

b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution: The problem can be made easier by using 1980 as the base year, that is, we choose the year 1980 as the year zero. This will mean that the year 2010 will correspond to year 30. Now we look at the information we have:

Year	Population
0 (1980)	24.5 million
30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y intercept occurs when $x = 0$, so b = 24.5

$$y = 0.32x + 24.5$$

b. Now to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$

$$y = 0.32x + 24.5$$

$$y = 0.32(45) + 24.5 = 38.9$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

1) The variable cost to manufacture a product is \$25 per item, and the fixed costs are \$1200. If x is the number of items manufactured and y is the cost, write the cost function.	2) It costs \$90 to rent a car driven 100 miles and \$140 for one driven 200 miles. If x is the number of miles driven and y the total cost of the rental, write the cost function.
3) The variable cost to manufacture an item is \$20 per item, and it costs a total of \$750 to produce 20 items. If x represents the number of items manufactured and y is the cost, write the cost function.	4) To manufacture 30 items, it costs \$2700, and to manufacture 50 items, it costs \$3200. If x represents the number of items manufactured and y the cost, write the cost function.
5) To manufacture 100 items, it costs \$32,000, and to manufacture 200 items, it costs \$40,000. If x is the number of items manufactured and y is the cost, write the cost function.	6) It costs \$1900 to manufacture 60 items, and the fixed costs are \$700. If x represents the number of items manufactured and y the cost, write the cost function.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

7) A person who weighs 150 pounds has 60 pounds of muscles; a person that weighs 180 pounds has 72 pounds of muscles. If x represents body weight and y is muscle weight, write an equation describing their relationship. Use this relationship to determine the muscle weight of a person that weighs 170 pounds.	8) A spring on a door stretches 6 inches if a force of 30 pounds is applied. It stretches 10 inches if a 50 pound force is applied. If x represents the number of inches stretched, and y is the force, write a linear equation describing the relationship. Use it to determine the amount of force required to stretch the spring 12 inches.
9). A male college student who is 64 inches tall weighs 110 pounds. Another student who is 74 inches tall weighs 180 pounds. Assuming the relationship between male students' heights (x), and weights (y) is linear, write a function to express weights in terms of heights, and use this function to predict the weight of a student who is 68 inches tall.	10) EZ Clean company has determined that if it spends \$30,000 on advertising, it can hope to sell 12,000 of its Minivacs a year, but if it spends \$50,000, it can sell 16,000. Write an equation that gives a relationship between the number of dollars spent on advertising (x) and the number of minivacs sold(y).
11) The freezing temperatures for water for Celsius and Fahrenheit scales are 0ºC and 32ºF. The boiling temperatures for water are 100 ºC and 212 ºF. Let C denote the temperature in Celsius and F in Fahrenheit. Write the conversion function from Celsius to Fahrenheit. Use the function to convert 25 ºC into ºF.	12) By reversing the coordinates in the previous problem, find a conversion function that converts Fahrenheit into Celsius, and use this conversion function to convert 72 ºF into an equivalent Celsius measure.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

13) California's population was 29.8 million in the year 1990, and 37.3 million in 2010. Assume that the population trend was and continues to be linear, write the population function. Use this function to predict the population in 2025. *Hint: Use 1990 as the base year (year 0); then 2010 and 2025 are years 20, and 35, respectively.)*	14) Use the population function for California in the previous problem to find the year in which the population will be 40 million people.
15) A college's enrollment was 13,200 students in the year 2000, and 15,000 students in 2015. Enrollment has followed a linear pattern. Write the function that models enrollment as a function of time. Use the function to find the college's enrollment in the year 2010. *Hint: Use year 2000 as the base year.*	16) If the college's enrollment continues to follow this pattern, in what year will the college have 16,000 students enrolled.
17) The cost of electricity in residential homes is a linear function of the amount of energy used. In Grove City, a home using 250 kilowatt hours (kwh) of electricity per month pays \$55. A home using 600 kwh per month pays \$118. Write the cost of electricity as a function of the amount used. Use the function to find the cost for a home using 400 kwh of electricity per month.	18) Find the level of electricity use that would correspond to a monthly cost of \$100.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

19) At ABC Co., sales revenue is \$170,000 when it spends \$5000 on advertising. Sales revenue is \$254,000 when \$12,000 is spent on advertising. a) Find a linear function for y = amount of sales revenue as a function of x = amount spent on advertising. b) Find revenue if \$10,000 is spent on advertising. c) Find the amount that should be spent on advertising to achieve \$200,000 in revenue.	20) For problem 19, explain the following: a. Explain what the slope of the line tells us about the effect on sales revenue of money spent on advertising. Be specific, explaining both the number and the sign of the slope in the context of this problem. b. Explain what the y intercept of the line tells us about the sales revenue in the context of this problem
21) Mugs Café sells 1000 cups of coffee per week if it does not advertise. For every \$50 spent in advertising per week, it sells an additional 150 cups of coffee. a) Find a linear function that gives y = number of cups of coffee sold per week x = amount spent on advertising per week. b) How many cups of coffee does Mugs Café expect to sell if \$100 per week is spent on advertising?	22) Party Sweets makes baked goods that can be ordered for special occasions. The price is \$24 to order one dozen (12 cupcakes) and \$9 for each additional 6 cupcakes. a) Find a linear function that gives the total price of a cupcake order as a function of the number of cupcakes ordered b) Find the price for an order of 5 dozen (60) cupcakes

1.5 More Applications

In this section, you will learn to:

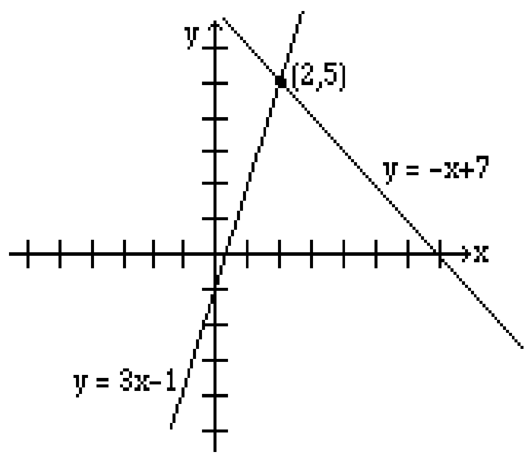
1. *Solve a linear system in two variables.*
2. *Find the equilibrium point when a demand and a supply equation are given.*
3. *Find the break-even point when the revenue and the cost functions are given.*

FINDING THE POINT OF INTERSECTION OF TWO LINES

In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

◆***Example 1*** Find the intersection of the line $y = 3x - 1$ and the line $y = -x + 7$.

Solution: We graph both lines on the same axes, as shown below, and read the solution (2, 5).



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x and y values for both lines are the same. So in order to find the intersection, we either let the x-values or the y-values equal.

If we were to solve the above example algebraically, it will be easier to let the y-values equal. Since $y = 3x - 1$ for the first line, and $y = -x + 7$ for the second line, by letting the y-values equal, we get

$$3x - 1 = -x + 7$$

$$4x = 8$$

$$x = 2$$

By substituting $x = 2$ in any of the two equations, we obtain $y = 5$.

Hence, the solution (2, 5).

A common algebraic method used to solve systems of equations is called the **elimination method**. The object is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable for can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we get the value of the other variable.

◆***Example 2*** Find the intersection of the lines $2x + y = 7$ and $3x - y = 3$ by the elimination method.

Solution: We add the left and right sides of the two equations.

$$\begin{array}{l} 2x + y = 7 \\ \underline{3x - y = 3} \\ 5x \quad = 10 \\ x = 2 \end{array}$$

Now we substitute $x = 2$ in any of the two equations and solve for y.

$$2(2) + y = 7$$

$$y = 3$$

Therefore, the solution is (2, 3).

◆***Example 3*** Solve the system of equations $x + 2y = 3$ and $2x + 3y = 4$ by the elimination method.

Solution: If we add the two equations, none of the variables are eliminated. But the variable x can be eliminated by multiplying the first equation by –2, and leaving the second equation unchanged.

$$\begin{array}{r} -2x - 4y = -6 \\ \underline{2x + 3y = 4} \\ -y = -2 \\ y = 2 \end{array}$$

Substituting $y = 2$ in $x + 2y = 3$, we get

$$x + 2(2) = 3$$

$$x = -1$$

Therefore, the solution is (–1, 2).

◆***Example 4*** Solve the system of equations $3x - 4y = 5$ and $4x - 5y = 6$.

Solution: This time, we multiply the first equation by – 4 and the second by 3 before adding. (The choice of numbers is not unique.)

$$\begin{array}{r} -12x + 16y = -20 \\ \underline{12x - 15y = 18} \\ y = -2 \end{array}$$

By substituting $y = -2$ in any one of the equations, we get $x = -1$.
Hence the solution is (–1, –2).

SUPPLY, DEMAND AND THE EQUILIBRIUM MARKET PRICE

In a free market economy the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices.

As the price of a product increases, its demand decreases and supply increases. On the other hand, as the price decreases the demand increases and supply decreases. The **equilibrium price** is reached when the demand equals the supply.

◆***Example 5*** The supply curve for a product is $y = 35x - 140$ and the demand curve for the same product is $y = -25x + 340$, where x is the price and y the number of items produced. Find the following.

a) How many items will be supplied at a price of \$10?

b) How many items will be demanded at a price of \$10?

c) Determine the equilibrium price.

d) How many items will be produced at the equilibrium price?

Solution: a) We substitute $x = 10$ in the supply equation, $y = 35x - 140$; the answer is $y = 35(10) - 140 = 210$ items are supplied if the price is \$10.

b) We substitute $x = 10$ in the demand equation, $y = -25x + 340$; the answer is $y = -25(10) + 340 = 90$ items are demanded if the price is \$10.

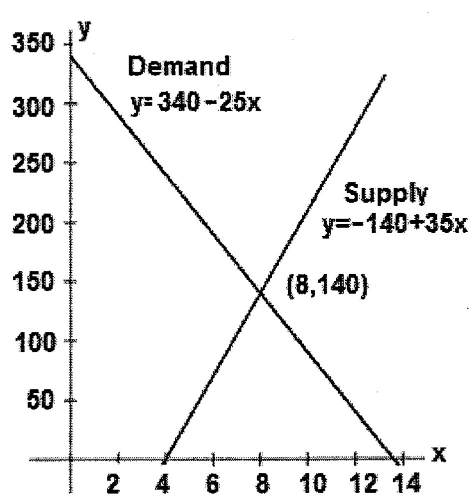
c) By letting the supply equal the demand, we get

$$35x - 140 = -25x + 340$$

$$60x = 480$$

$$x = \$8$$

d) We substitute $x = 8$ in either the supply or the demand equation; we get $y = 140$.



The graph shows the intersection of the supply and the demand functions and their point of intersection, (8, 140).

Interpretation:
At equilibrium, the price is \$8 per item, and 140 items are produced by suppliers and purchased by consumers.

BREAK-EVEN POINT

In a business, the profit is generated by selling products.

- If a company sells x number of items at a price P, then the **revenue R** is the price multiplied by number of items sold: R = P • x.
- The **production costs C** are the sum of the variable costs and the fixed costs, and are often written as C = mx + b, where x is the number of items manufactured.
  - The slope m is the called marginal cost and represents the cost to produce one additional item or unit.
  - The variable cost, mx, depends on how much is being produced
  - The fixed cost b is constant; it does not change no matter how much is produced.
- **Profit** is equal to Revenue minus Cost: Profit = R – C

A company makes a profit if the revenue is greater than the cost. There is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the **break-even point**. At the break-even point, profit is 0.

◆***Example 6*** If the revenue function of a product is R = 5x and the cost function is y = 3x + 12, find the following.

a) If 4 items are produced, what will the revenue be?

b) What is the cost of producing 4 items?

c) How many items should be produced to break even?

d) What will be the revenue and the cost at the break-even point?

Solution: a) We substitute x = 4 in the revenue equation R = 5x, and the answer is R = 20.

b) We substitute x = 4 in the cost equation C = 3x + 12, and the answer is C = 24.

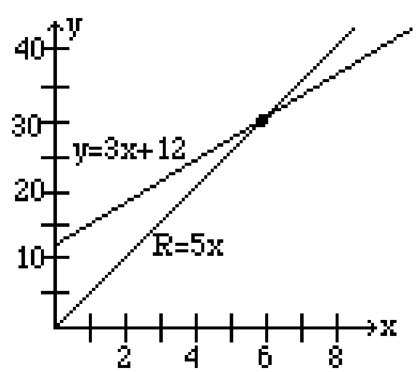
c) By letting the revenue equal the cost, we get

$$5x = 3x + 12$$

$$x = 6$$

d) Substitute x = 6 in either the revenue or the cost equation: we get R = C = 30.

The graph below shows the intersection of the revenue and cost functions and their point of intersection, (6, 30).



SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

1) Solve for x and y. $y = 3x + 4$ $y = 5x - 2$	2) Solve for x and y. $2x - 3y = 4$ $3x - 4y = 5$
3) The supply and demand curves for a product are: Supply $y = 2000x - 6500$ Demand $y = -1000x + 28000$, where x is price and y is the number of items. At what price will supply equal demand and how many items will be produced at that price?	4) The supply and demand curves for a product are Supply $y = 300x - 18000$ and Demand $y = -100x + 14000$, where x is price and y is the number of items. At what price will supply equal demand, and how many items will be produced at that price?

5) A car rental company offers two plans for one way rentals.
Plan I charges $36 per day and 17 cents per mile. Plan II charges $24 per day and 25 cents per mile.
a. If you were to drive 300 miles in a day, which plan is better?
b. For what mileage are both rates equal?

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

6) A demand curve for a product is the number of items the consumer will buy at different prices. At a price of \$2 a store can sell 2400 of a particular type of toy truck. At a price of \$8 the store can sell 600 such trucks. If x represents the price of trucks and y the number of items sold, write an equation for the demand curve.	7) A supply curve for a product is the number of items that can be made available at different prices. A manufacturer of toy trucks can supply 2000 trucks if they are sold for \$8 each; it can supply only 400 trucks if they are sold for \$4 each. If x is the price and y the number of items, write an equation for the supply curve.
8) The equilibrium price is the price where the supply equals the demand. From the demand and supply curves obtained in the previous two problems, find the equilibrium price, and determine the number of items that can be sold at that price.	9) A break-even point is the intersection of the cost function and the revenue function, that is, where total cost equals revenue, and profit is zero. Mrs. Jones Cookies Store's cost and revenue, in dollars, for x number of cookies is given by $C = .05x + 3000$ and $R = .80x$. Find the number of cookies that must be sold to break even.

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

10) A company's revenue and cost in dollars are given by R = 225x and C = 75x + 6000, where x is the number of items. Find the number of items that must be produced to break-even.	11) A firm producing socks has a fixed cost of \$20,000 and variable cost of \$2 per pair of socks. Let x = the number of pairs of socks. Find the break-even point if the socks sell for \$4.50 per pair.
12) Whackemhard Sports is planning to introduce a new line of tennis rackets. The fixed costs for the new line are \$25,000 and the variable cost of producing each racket is \$60. x is the number of rackets; y is in dollars. If the racket sells for \$80, how many rackets must be sold in order to break even?	13) It costs \$1,200 to produce 50 pounds of a chemical and it costs \$2,200 to produce 150 pounds. The chemical sells for \$15 per pound x is the amount of chemical; y is in dollars. a. Find the cost function. b. What is the fixed cost? c. How many pounds must be sold to break even? d. Find the cost and revenue at the break-even point.

SECTION 1.6 PROBLEM SET: CHAPTER REVIEW

1) Find an equation of the x-axis.

2) Find the slope of the line whose equation is $2x + 3y = 6$.

3) Find the slope of the line whose equation is $y = -3x + 5$.

4) Find both the x and y intercepts of the line $3x - 2y = 12$.

5) Find an equation of the line whose slope is 3 and y-intercept 5.

6) Find an equation of the line whose x-intercept is 2 and y-intercept 3.

7) Find an equation of the line that has slope 3 and passes through the point (2, 15).

8) Find an equation of the line that has slope –3/2 and passes through the point (4, 3).

9) Find an equation of the line that passes through the points (0, 32) and (100, 212).

10) Find an equation of the line that passes through the point (2, 5) and is parallel to the line $y = 3x + 4$.

11) Find the point of intersection of the lines $2x - 3y = 9$ and $3x + 4y = 5$.

12) Is the point (3, – 2) on the line $5x - 2y = 11$?

13) Find two points on the line given by the parametric equations, $x = 2 + 3t$, $y = 1 - 2t$.

14) Find two points on the line $2x - 6 = 0$.

15) Graph the line $2x - 3y + 6 = 0$.

16) Graph the line $y = -2x + 3$.

17) A female college student who is 60 inches tall weighs 100 pounds. Another female student who is 66 inches tall weighs 124 pounds. Assume the relationship between the female students' weights and heights is linear. Find an equation for weight as a function of height. Use this relationship to predict the weight of a female student who is 70 inches tall.

18) In deep-sea diving, the pressure exerted by water plays a great role in designing underwater equipment. If at a depth of 10 feet there is a pressure of 21 lb/in^2, and at a depth of 50 ft there is a pressure of 75 lb/in^2, write a linear equation giving a relationship between depth and pressure. Use this relationship to predict pressure at a depth of 100 ft.

19) The variable cost to manufacture an item is \$30 per item; the fixed costs are \$2750. Find the cost function.

20) The variable cost to manufacture an item is \$10 per item, and it costs \$2,500 to produce 100 items. Write the cost function, and use this function to estimate the cost of manufacturing 300 items.

21) It costs \$2,700 to manufacture 100 items of a product, and \$4,200 to manufacture 200 items. x= the number of items; y= cost. Find the cost function; use it to predict the cost to produce 1000 items.

22) In 1990, the average house in Emerald City cost \$280,000 and in 2007 the same house cost \$365,000. Assuming a linear relationship, write an equation that will give the price of the house in any year, and use this equation to predict the price of a similar house in the year 2020.

23) The population of Mexico in 1995 was 95.4 million and in 2010 it was 117.9 million. Assuming a linear relationship, write an equation that will give the population of Mexico in any year, and use this equation to predict the population of Mexico in the year 2025.

SECTION 1.6 PROBLEM SET: CHAPTER REVIEW

24) At Nuts for Soup Lunch Bar, they sell 150 bowls of soup if the high temperature for the day is 40 ºF. For every 5 ºF increase in high temperature for the day, they sell 10 fewer bowls of soup.
   a. Assuming a linear relationship, write an equation that will give y = the number of bowls of soup sold as a function of x = the daily high temperature.
   b. How many bowls of soup are sold when the temperature is 75 ºF?
   c. What is the temperature when 100 bowls of soup are sold?

25) Two-hundred items are demanded at a price of $5, and 300 items are demanded at a price of $3. If x represents the price, and y the number of items, write the demand function.

26) A supply curve for a product is the number of items of the product that can be made available at different prices. A doll manufacturer can supply 2000 dolls if the dolls are sold for $30 each, but he can supply only 400 dolls if the dolls are sold for $10 each. If x represents the price of dolls and y the number of items, write an equation for the supply curve.

27) Suppose you are trying to decide on a price for your latest creation - a coffee mug that never tips. Through a survey, you have determined that at a price of $2, you can sell 2100 mugs, but at a price of $12 you can only sell 100 mugs. Furthermore, your supplier can supply you 3100 mugs if you charge your customers $12, but only 100 mugs if you charge $2. What price should you charge so that the supply equals demand, and at that price how many coffee mugs will you be able to sell?

28) A car rental company offers two plans. Plan I charges $16 a day and 25 cents a mile, while Plan II charges $45 a day but no charge for miles. If you were to drive 200 miles in a day, which plan is better? For what mileage are both rates the same?

29) The supply curve for a product is y = 250x – 1000. The demand curve for the same product is y = – 350x + 8,000, where x is the price and y the number of items produced. Find the following.
   a) At a price of $10, how many items will be in demand?
   b) At what price will 4,000 items be supplied?
   c) What is the equilibrium price for this product?
   d) How many items will be manufactured at the equilibrium price?

30) The supply curve for a product is y = 625x – 600 and the demand curve for the same product is y = – 125x + 8,400, where x is the price and y the number of items produced.
   Find the equilibrium price and determine the number of items that will be produced at that price.

31) Both Jenny and Masur work in the sales department for Sports Supply. Jenny gets paid $120 per day plus 4% commission on the sales. Masur gets paid $132 per day plus 8% commission on the sales in excess of $1,000. For what sales amount would they both earn the same daily amounts?

32) A company's revenue and cost in dollars are given by R = 25x and C = 10x + 9,000, where x represents the number of items. Find the number of items that must be produced to break-even.

33) A firm producing a certain type of CFL lightbulb has fixed costs of $6,800, and a variable cost of $2.30 per bulb. The bulbs sell for $4 each. How many bulbs must be produced to break-even?

34) A company producing tire pressure gauges has fixed costs of $7,500, and variable cost of $1.50 cents per item. If the gauges sell for $4.50, how many must be produced to break-even?

35) A company is introducing a new cordless travel shaver before the Christmas holidays. It hopes to sell 15,000 of these shavers in December. The variable cost is $11 per item and the fixed costs $100,000. If the shavers sell for $19 each, how many must be produced and sold to break-even?

CHAPTER 2

Chapter 2: Matrices

In this chapter, you will learn to:

1. *Do matrix operations.*
2. *Solve linear systems using the Gauss-Jordan method.*
3. *Solve linear systems using the matrix inverse method.*
4. *Do application problems.*

2.1 Introduction to Matrices

In this section you will learn to:

1. *Add and subtract matrices.*
2. *Multiply a matrix by a scalar.*
3. *Multiply two matrices.*

A matrix is a 2 dimensional array of numbers arranged in rows and columns.

Matrices provide a method of organizing, storing, and working with mathematical information. Matrices have an abundance of applications and use in the real world.

Matrices provide a useful tool for working with models based on systems of linear equations. We'll use matrices in sections 2.2, 2.3, and 2.4 to solve systems of linear equations with several variables in this chapter.

Matrices are used in encryption, which we will explore in section 2.5 and in economic modelling, explored in section 2.6.

We use matrices again in chapter 4, in optimization problems such as maximizing profit or revenue, or minimizing cost. Matrices are used in business for scheduling, routing transportation and shipments, and managing inventory.

Just about any application that collects and manages data can apply matrices. Use of matrices has grown as the availability of data in many areas of life and business has increased. They are important tools for organizing data and solving problems in all fields of science, from physics and chemistry, to biology and genetics, to meteorology, and economics. In computer science, matrix mathematics lies behind animation of images in movies and video games.

Computer science analyzes diagrams of networks to understand how things are connected to each other, such as relationships between people on a social website, and relationships between results in line search and how people link from one website to another. The mathematics to work with network diagrams comprise the field of "graph theory"; it relies on matrices to organize the information in the graphs that diagram connections and associations in a network. For example, if you use Facebook or Linked-In, or other social media sites, these sites use network graphs and matrices to organize your relationships with other users.

INTRODUCTION TO MATRICES

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

◆***Example 1*** Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.

	2014		2015	
	CHAIRS	TABLES	CHAIRS	TABLES
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

a) Represent the production for the years 2014 and 2015 as the matrices A and B.

b) Find the difference in sales between the years 2014 and 2015.

c) The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?

Solution: a) The matrices are as follows: $A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix}$ $B = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$

b) We are looking for the matrix B – A. When two matrices have the same number of rows and columns, the matrices can be added or subtracted entry by entry. Therefore, we get

$$B - A = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

c) We would like a matrix that is twice the matrix of 2014, i.e., 2A.

Whenever a matrix is multiplied by a number, each entry is multiplied by the number.

$$2A = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices.

The numbers in a matrix are called the **entries** or the **elements** of a matrix.

Whenever we talk about a matrix, we need to know the **size** or the **dimension** of the matrix. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a “3 by 4 matrix”, we are saying that it has 3 rows and 4 columns. The rows are always mentioned first and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A=\begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix A has dimensions 3×4 Matrix B has dimensions 4×3

A matrix that has the same number of rows as columns is called a **square matrix**.

A matrix with all entries zero is called a **zero matrix.**

A square matrix with 1's along the main diagonal and zeros everywhere else, is called an **identity matrix**. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a row matrix or a **row vector,** and a matrix with only one column is called a column matrix or a **column vector**.

Two matrices are **equal** if they have the same size and the corresponding entries are equal.

We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

MATRIX ADDITION AND SUBTRACTION

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

◆***Example 2*** Given the matrices A, B, C and D, below

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \quad D = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find, if possible. a) A + B b) C – D c) A + D.

Solution: As we mentioned earlier, matrix addition and subtraction involves performing these operations entry by entry.

a) We add each element of A to the corresponding entry of B.

$$A + B = \begin{bmatrix} 3 & 1 & 7 \\ 4 & 7 & 3 \\ 8 & 6 & 4 \end{bmatrix}$$

b) Just like the problem above, we perform the subtraction entry by entry.

$$C - D = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

c) The sum A + D cannot be found because the two matrices have different sizes.

Note: Two matrices can only be added or subtracted if they have the same dimension.

MULTIPLYING A MATRIX BY A SCALAR

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar. We can consider scalar multiplication as multiplying a number and a matrix to obtain a new matrix as the product.

◆***Example 3*** Given the matrix A and C in the example above, find 2A and – 3C.

Solution: To find 2A, we multiply each entry of matrix A by 2, and to find –3C, we multiply each entry of C by –3. The results are given below.

a) We multiply each entry of A by 2.

$$2A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 6 & 2 \\ 10 & 0 & 6 \end{bmatrix}$$

b) We multiply each entry of C by –3.

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

MULTIPLICATION OF TWO MATRICES

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner. We first begin by finding a product of a row matrix and a column matrix.

◆***Example 4*** Given A = $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, find the product AB.

Solution: The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum.

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= [\ (2a + 3b + 4c)]$$

Note that AB is a 1×1 matrix, and its only entry is 2a + 3b + 4c.

◆***Example 5*** Given A = $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, find the product AB.

Solution: Again, we multiply the corresponding entries and add.

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

$$= [\ 2\cdot 5 + 3\cdot 6 + 4\cdot 7\ ]$$

$$= [\ 10 + 18 + 28\ ]$$

$$= [\ 56\ ]$$

Note: In order for a product of a row matrix and a column matrix to exist, the number of entries in the row matrix must be the same as the number of entries in the column matrix.

◆***Example 6*** Given A = $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.

Solution: We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix.

$$AB = \begin{bmatrix} 2\cdot 5 + 3\cdot 6 + 4\cdot 7 & 2\cdot 3 + 3\cdot 4 + 4\cdot 5 \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions, if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

◆***Example 7*** Given A = $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.

Solution: This time we are multiplying two rows of the matrix A with two columns of the matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that the matrix A has one more row.

We multiply the first row of the matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 5+3\cdot 6+4\cdot 7 & 2\cdot 3+3\cdot 4+4\cdot 5 \\ 1\cdot 5+2\cdot 6+3\cdot 7 & 1\cdot 3+2\cdot 4+3\cdot 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

◆***Example 8*** Find, if possible: a) EF b) FE c) FH d) GH e) HG

E = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$ F = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ G = $\begin{bmatrix} 4 & 1 \end{bmatrix}$ H = $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Solution: a) To find EF, we multiply the first row [1 2] of E with the columns $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ of the matrix F, and then repeat the process by multiplying the other two rows of E with these columns of F. The result is as follows:

$$EF = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+2\cdot 3 & 1\cdot -1+2\cdot 2 \\ 4\cdot 2+2\cdot 3 & 4\cdot -1+2\cdot 2 \\ 3\cdot 2+1\cdot 3 & 3\cdot -1+1\cdot 2 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 14 & 0 \\ 9 & -1 \end{bmatrix}$$

b) Product FE is not possible because the matrix F has two entries in each row, while the matrix E has three entries in each column. In other words, the matrix F has two columns, while the matrix E has three rows.

c) $FH = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\cdot -3+-1\cdot -1 \\ 3\cdot -3+2\cdot -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$

d) $GH = \begin{bmatrix} 4 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = [\, 4\cdot -3 + 1\cdot -1 \,] = [-13]$

e) $HG = \begin{bmatrix} -3 \\ -1 \end{bmatrix}\begin{bmatrix} 4 & 1 \end{bmatrix} = \begin{bmatrix} -3\cdot 4 & -3\cdot 1 \\ -1\cdot 4 & -1\cdot 1 \end{bmatrix} = \begin{bmatrix} -12 & -3 \\ -4 & -1 \end{bmatrix}$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

In order for product AB to exist:

- **the number of columns of A must equal the number of rows of B**
- **if matrix A has dimension $m \times n$ and matrix B has dimension $n \times p$, then the product AB will be a matrix with dimension $m \times p$.**

Matrix multiplication is not commutative: if both matrix products AB and BA exist, most of the time AB will not equal BA.

◆***Example 9*** Given matrices R, S, and T below, find 2RS – 3ST.

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \quad T = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution: We multiply the matrices R and S.

$$RS = \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix}$$

$$2RS = 2\begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix}$$

$$ST = \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

$$3ST = 3\begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix}$$

$$\text{Thus } 2RS - 3ST = \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix} = \begin{bmatrix} 13 & 6 & 14 \\ 73 & -15 & 12 \\ 71 & -45 & -2 \end{bmatrix}$$

◆***Example 10*** Given matrix $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, find F^2.

Solution: F^2 is found by multiplying matrix F by itself, using matrix multiplication.

$$F^2 = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + (-1)\cdot 3 & 2\cdot(-1) + (-1)\cdot 2 \\ 3\cdot 2 + 2\cdot 3 & 3\cdot(-1) + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$$

Note that F^2 is not found by squaring each entry of matrix F.
The process of raising a matrix to a power, such as finding F^2, is only possible if the matrix is a square matrix.

USING MATRICES TO REPRESENT A SYSTEM OF LINEAR EQUATIONS

In this chapter, we will be using matrices to solve linear systems. In section 2.4, we will be asked to express linear systems as the **matrix equation AX = B**, where A, X, and B are matrices.

- Matrix A is called the **coefficient matrix.**
- Matrix X is a matrix with 1 column that contains the variables.
- Matrix B is a matrix with 1 column that contains the constants.

◆***Example 11*** Verify that the system of two linear equations with two unknowns:

$$ax + by = h$$
$$cx + dy = k$$

can be written as AX = B, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} h \\ k \end{bmatrix}$$

Solution: If we multiply the matrices A and X, we get

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If AX = B then

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$ax + by = h$$
$$cx + dy = k$$

◆***Example 12*** Express the following system as a matrix equation in the form AX = B.

$$2x + 3y - 4z = 5$$
$$3x + 4y - 5z = 6$$
$$5x \qquad - 6z = 7$$

Solution: This system of equations can be expressed in the form AX = B as shown below.

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

SECTION 2.1 PROBLEM SET: INTRODUCTION TO MATRICES

A vendor sells hot dogs and corn dogs at three different locations. His total sales(in hundreds) for January and February from the three locations are given in the table below.

	JANUARY		FEBRUARY	
	HOT DOGS	CORN DOGS	HOT DOGS	CORN DOGS
PLACE I	10	8	8	7
PLACE II	8	6	6	7
PLACE III	6	4	6	5

Represent these tables as 3×2 matrices J and F, and answer problems 1 - 5.

1) Determine total sales for the two months, that is, find J + F.

2) Find the difference in sales, J – F.

3) If hot dogs sell for \$3 and corn dogs for \$2, find the revenue from the sale of hot dogs and corn dogs. *Hint: Let P be a 2×1 matrix. Find (J + F)P.*

4) If March sales will be up from February by 10%, 15%, and 20% at Place I, Place II, and Place III, respectively, find the expected number of hot dogs and corn dogs to be sold in March. *Hint: Let R be a 1×3 matrix with entries 1.10, 1.15, and 1.20. Find M = RF.*

5) Hots dogs sell for \$3 and corn dogs sell for \$2. Using matrix M that predicts the number of hot dogs and corn dogs expected to be sold in March from problem (4), find the 1x1 matrix that predicts total revenue in March. *Hint: Use 2x1 price matrix P from problem (3) and find MP.*

SECTION 2.1 PROBLEM SET: INTRODUCTION TO MATRICES

Determine the sums and products in problems 6-13. Given the matrices A, B, C, and D as follows:

$$A = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}$$

6) 3A – 2B	7) AB
8) BA	9) AB + BA
10) A^2	11) 2BC
12) 2CD + 3AB	13) A^2B

SECTION 2.1 PROBLEM SET: INTRODUCTION TO MATRICES

14) Let $E = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ and $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find EF.	15) Let $E = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ and $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find FE.
16) Let $G = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ $H = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find GH.	17) Let $G = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ $H = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Explain why the product HG does not exist.

Express the following systems as AX = B, where A, X, and B are matrices.

18) $4x - 5y = 6$ $5x - 6y = 7$	19) $x - 2y + 2z = 3$ $x - 3y + 4z = 7$ $x - 2y - 3z = -12$
20) $2x + 3z = 17$ $3x - 2y = 10$ $5y + 2z = 11$	21) $x + 2y + 3z + 2w = 14$ $x - 2y - z = -5$ $y - 2z + 4w = 9$ $x + 3z + 3w = 15$

2.2 Systems of Linear Equations; Gauss-Jordan Method

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix

2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the **augmented matrix**, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a **row operation**.

◆***Example 1*** Write the following system as an augmented matrix.

$$2x + 3y - 4z = 5$$
$$3x + 4y - 5z = -6$$
$$4x + 5y - 6z = 7$$

Solution: We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as AX = B, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as [A | B]. It is clear that all of the information is maintained in this matrix form, and only the letters x, y and z are missing. A student may choose to write x, y and z on top of the first three columns to help ease the transition.

◆***Example 2*** For the following augmented matrix, write the system of equations it represents.

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 2 & 0 & -3 & -5 \\ 3 & 2 & -3 & -1 \end{array}\right]$$

Solution: The system is readily obtained as below.

$$x + 3y - 5z = 2$$
$$2x \qquad - 3z = -5$$
$$3x + 2y - 3z = -1$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the **reduced row echelon form**. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

◆***Example 3*** Solve the following system by the elimination method.

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution: We multiply the first equation by – 3, and add it to the second equation.

$$\begin{array}{r} -3x - 9y = -21 \\ \underline{3x + 4y = 11} \\ -5y = -10 \end{array}$$

By doing this we transformed our original system into an equivalent system:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

We divide the second equation by – 5, and we get the next equivalent system.

$$\begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Now we multiply the second equation by – 3 and add to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

◆***Example 4*** Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution: The augmented matrix for the system is as follows.

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 3 & 4 & | & 11 \end{bmatrix} \qquad \begin{bmatrix} x + 3y = 7 \\ 3x + 4y = 11 \end{bmatrix}$$

We multiply the first row by – 3, and add to the second row.

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 0 & -5 & | & -10 \end{bmatrix} \qquad \begin{bmatrix} x + 3y = 7 \\ -5y = -10 \end{bmatrix}$$

We divide the second row by – 5, we get,

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 0 & 1 & | & 2 \end{bmatrix} \qquad \begin{bmatrix} x + 3y = 7 \\ y = 2 \end{bmatrix}$$

Finally, we multiply the second row by – 3 and add to the first row, and we get,

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \end{bmatrix} \qquad \begin{bmatrix} x = 1 \\ y = 2 \end{bmatrix}$$

Now we list the three row operations the Gauss-Jordan method employs.

Row Operations

1. **Any two rows in the augmented matrix may be interchanged.**
2. **Any row may be multiplied by a non-zero constant.**
3. **A constant multiple of a row may be added to another row.**

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

The first row operation states that if any two rows of a system are interchanged, the new system obtained has the same solution as the old one. Let us look at an example in two equations with two unknowns. Consider the system

$$x + 3y = 7$$
$$3x + 4y = 11$$

We interchange the rows, and we get,

$$3x + 4y = 11$$
$$x + 3y = 7$$

Clearly, this system has the same solution as the one above.

The second operation states that if a row is multiplied by any non-zero constant, the new system obtained has the same solution as the old one. Consider the above system again,

$$x + 3y = 7$$
$$3x + 4y = 11$$

We multiply the first row by –3, we get,

$$-3x - 9y = -21$$
$$3x + 4y = 11$$

Again, it is obvious that this new system has the same solution as the original.

The third row operation states that any constant multiple of one row added to another preserves the solution. Consider our system,

$$x + 3y = 7$$
$$3x + 4y = 11$$

If we multiply the first row by –3, and add it to the second row, we get,

$$x + 3y = 7$$
$$-5y = -10$$

And once again, the same solution is maintained.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations.

As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the **reduced row echelon form**.

A matrix is in the **reduced row echelon form** if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom.

We state the Gauss-Jordan method as follows.

Gauss-Jordan Method

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

◆***Example 5*** Solve the following system by the Gauss-Jordan method.

$$\begin{aligned} 2x + y + 2z &= 10 \\ x + 2y + z &= 8 \\ 3x + y - z &= 2 \end{aligned}$$

Solution: We write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \qquad \text{we interchanged row 1(R1) and row 2(R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by - 2 and add it to the second row. We get,

$$\begin{bmatrix} 1 & 2 & 1 & | & 8 \\ 0 & -3 & 0 & | & -6 \\ 3 & 1 & -1 & | & 2 \end{bmatrix} \quad -2R1 + R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by - 3 and add it to the third row. We get,

$$\begin{bmatrix} 1 & 2 & 1 & | & 8 \\ 0 & -3 & 0 & | & -6 \\ 0 & -5 & -4 & | & -22 \end{bmatrix} \quad -3R1 + R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need to make this entry(–3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by –3.

$$\begin{bmatrix} 1 & 2 & 1 & | & 8 \\ 0 & 1 & 0 & | & 2 \\ 0 & -5 & -4 & | & -22 \end{bmatrix} \quad R2 \div (-3)$$

Next, we make all other entries zeros in the second column.

$$\begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & -4 & | & -12 \end{bmatrix} \quad -2R2 + R1 \text{ and } 5R2 + R3$$

We make the last diagonal entry a 1, by dividing row 3 by – 4.

$$\begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R3 \div (-4)$$

Finally, we make all other entries zeros in column 3.

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad -R3 + R1$$

Clearly, the solution reads $x = 1$, $y = 2$, and $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called **pivoting**.

The number that is made a 1 is called the **pivot element,** and the row that contains the pivot element is called the **pivot row**.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the **target row**.

SECTION 2.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

1) $x + 3y = 1$ $2x - 5y = 13$	2) $x - y - z = -1$ $x - 3y + 2z = 7$ $2x - y + z = 3$
3) $x + 2y + 3z = 9$ $3x + 4y + z = 5$ $2x - y + 2z = 11$	4) $x + 2y = 0$ $y + z = 3$ $x + 3z = 14$

SECTION 2.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

5) Two apples and four bananas cost \$2.00 and three apples and five bananas cost \$2.70. Find the price of each.

6) A bowl of corn flakes, a cup of milk, and an egg provide 16 grams of protein. A cup of milk and two eggs provide 21 grams of protein. Two bowls of corn flakes with two cups of milk provide 16 grams of protein. How much protein is provided by one unit of food?

7)
$$\begin{aligned} x + 2y &= 10 \\ y + z &= 5 \\ z + w &= 3 \\ x + w &= 5 \end{aligned}$$

8)
$$\begin{aligned} x + w &= 6 \\ 2x + y + w &= 16 \\ x - 2z &= 0 \\ z + w &= 5 \end{aligned}$$

2.3 Systems of Linear Equations – Special Cases

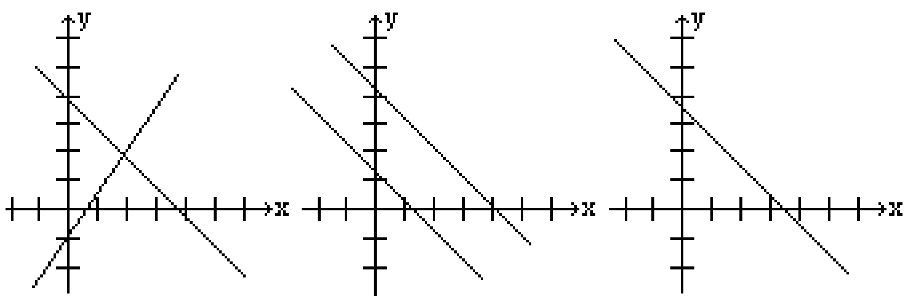
In this section you will learn to:

1. *Determine the linear systems that have no solution.*
2. *Solve the linear systems that have infinitely many solutions.*

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an **independent system**.
2. The lines are parallel, so they do not intersect. This is called an **inconsistent system**.
3. The lines coincide; they intersect at infinitely many points. This is a **dependent system**.

The figures below show all three cases.



Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

◆***Example 1*** Solve the following system of equations:

$$\begin{aligned} x + y &= 7 \\ x + y &= 9 \end{aligned}$$

Solution: Let us use the Gauss-Jordan method to solve this system.   The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 1 & 9 \end{array}\right] \qquad \left[\begin{array}{c} x + y = 7 \\ x + y = 9 \end{array}\right]$$

If we multiply the first row by – 1 and add to the second row, we get

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 2 \end{array}\right] \qquad \left[\begin{array}{c} x + y = 7 \\ 0x + 0y = 2 \end{array}\right]$$

Since 0 cannot equal 2, the last equation cannot be true for any choices of x and y.

Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

◆***Example 2*** Solve the following system of equations.

$$\begin{aligned} 2x + 3y - 4z &= 7 \\ 3x + 4y - 2z &= 9 \\ 5x + 7y - 6z &= 20 \end{aligned}$$

Solution: We enter the following augmented matrix in the calculator.

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right]$$

Now by pressing the key to obtain the reduced row-echelon form, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row states that $0x + 0y + 0z = 1$. But the left side of the equation is equal to 0. So this last row states $0 = 1$, which is a contradiction, a false statement.

This bottom row indicates that the system is inconsistent; therefore, there is no solution.

◆***Example 3*** Solve the following system of equations.

$$\begin{aligned} x + y &= 7 \\ x + y &= 7 \end{aligned}$$

Solution: The problem clearly asks for the intersection of two lines that are the same; that is, the lines coincide. This means the lines intersect at an infinite number of points.

A few intersection points are listed as follows: (3, 4), (5, 2), (–1, 8), (–6, 13) etc. However, when a system has an infinite number of solutions, the solution is often expressed in the parametric form. This can be accomplished by assigning an arbitrary constant, t, to one of the variables, and then solving for the remaining variables. Therefore, if we let $y = t$, then $x = 7 - t$. Or we can say all ordered pairs of the form $(7 - t, t)$ satisfy the given system of equations.

Alternatively, while solving the Gauss-Jordan method, we will get the reduced row-echelon form given below.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

The row of all zeros, can simply be ignored. This row says $0x + 0y = 0$; it provides no further information about the values of x and y that solve this system.

This leaves us with only one equation but two variables. And whenever there are more variables than the equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as above.

Parametric Solution: $x = 7 - t, y = t$.

◆***Example 4*** Solve the following system of equations.

$$\begin{aligned} x + y + z &= 2 \\ 2x + y - z &= 3 \\ 3x + 2y \quad &= 5 \end{aligned}$$

Solution: The augmented matrix and the reduced row-echelon form are given below.

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 2 & 1 & -1 & | & 3 \\ 3 & 2 & 0 & | & 5 \end{bmatrix}$$ Augmented Matrix for this system

$$\begin{bmatrix} 1 & 0 & -2 & | & 1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$ Reduced Row Echelon Form

Since the last equation dropped out, we are left with two equations and three variables. This means the system has infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$.

The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$.

And now if we let $z = t$, the parametric solution is expressed as follows:

Parametric Solution: $x = 1 + 2t$, $y = 1 - 3t$, $z = t$.

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- if we let $t = 2$, we have the solution $x = 5$, $y = -5$, $z = 2$: (5, –5, 2)
- if we let $t = 0$, we have the solution $x = 1$, $y = 1$, $z = 0$: (1, 1, 0).

◆***Example 5*** Solve the following system of equations.

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + 4y - 6z &= 10 \\ 3x + 6y - 9z &= 15 \end{aligned}$$

Solution: The reduced row-echelon form is given below.

$$\begin{bmatrix} 1 & 2 & -3 & | & 5 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$ This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.

If we let $z = t$ and let $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$.

We rewrite the **parametric solution : $x = 5 - 2s + 3t$, $y = s$, $z = t$.**

We summarize our discussion in the following table.

1. If any row of the reduced row-echelon form of the matrix gives a false statement such as 0 = 1, the system is inconsistent and has no solution.
2. If the reduced row echelon form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.
    a. If a system has an infinite number of solutions, the solution must be expressed in the parametric form.
    b. The number of arbitrary parameters equals the number of variables minus the number of equations.

SECTION 2.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS – SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

1) $2x + 6y = 8$ $x + 3y = 4$	2) The sum of digits of a two digit number is 9. The sum of the number and the number obtained by interchanging the digits is 99. Find the number.
3) $2x - y = 10$ $-4x + 2y = 15$	4) $x + y + z = 6$ $3x + 2y + z = 14$ $4x + 3y + 2z = 20$
5) $x + 2y - 4z = 1$ $2x - 3y + 8z = 9$	6) Jessica has a collection of 15 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all three solutions.

SECTION 2.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS – SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

7) A company is analyzing sales reports for three products: products X, Y, Z. One report shows that a combined total of 20,000 of items X, Y, and Z were sold. Another report shows that the sum of the number of item Z sold and twice the number of item X sold equals 10,000. Also item X has 5,000 more items sold than item Y.
Are these reports consistent?

8)
$$x + y + 2z = 0$$
$$x + 2y + z = 0$$
$$2x + 3y + 3z = 0$$

9) Find three solutions to the following system of equations.
$$x + 2y + z = 12$$
$$y = 3$$

10)
$$x + 2y = 5$$
$$2x + 4y = k$$
For what values of k does this system of equations have
a) No solution?

b) Infinitely many solutions?

11) $x + 3y - z = 5$

12) Why is it not possible for a linear system to have exactly two solutions? Explain geometrically.

2.4 Inverse Matrices

In this section you will learn to:

1. *Find the inverse of a matrix, if it exists.*
2. *Use inverses to solve linear systems.*

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition of an Inverse: An $n \times n$ matrix has an inverse if there exists a matrix B such that $AB = BA = I_n$, where I_n is an $n \times n$ identity matrix. The inverse of a matrix A, if it exists, is denoted by the symbol A^{-1}.

◆***Example 1*** Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution: The matrices are inverses if the product AB and BA both equal the identity matrix of dimension 2×2: I_2,

$$AB = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Clearly that is the case; therefore, the matrices A and B are inverses of each other.

◆***Example 2*** Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution: Suppose A has an inverse, and it is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = I_2$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a + c & 3b + d \\ 5a + 2c & 5b + 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns:

$3a + c = 1$ $\quad$ $3b + d = 0$

$5a + 2c = 0$ $\quad$ $5b + 2d = 1$

Solving this system, we get: $a = 2$ $\quad b = -1$ $\quad c = -5$ $\quad d = 3$

Therefore, the inverse of the matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$3a + c = 1 \qquad 3b + d = 0$$
$$5a + 2c = 0 \qquad 5b + 2d = 1$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\begin{bmatrix} 3 & 1 & | & 1 \\ 5 & 2 & | & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 1 & | & 0 \\ 5 & 2 & | & 1 \end{bmatrix}$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right hand columns are grouped together to form one augmented matrix as below.

$$\begin{bmatrix} 3 & 1 & | & 1 & 0 \\ 5 & 2 & | & 0 & 1 \end{bmatrix}$$

And solving this system, we get

$$\begin{bmatrix} 1 & 0 & | & 2 & -1 \\ 0 & 1 & | & -5 & 3 \end{bmatrix}$$

The matrix on the right side of the vertical line is the A^{-1} matrix.

What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix. We list the steps, as follows:

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[\ A \mid I_n\ ]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[\ I_n \mid B]$, then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

◆***Example 3*** Given the matrix A, find its inverse: $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Solution: We write the augmented matrix as follows.

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 0 & | & 0 & 1 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

We will reduce this matrix using the Gauss-Jordan method.

Multiplying the first row by –2 and adding it to the second row, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by –2. The result is

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) Add the second row to first, 2) Add -5 times the second row to the third. And we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 results in

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Multiply the third row by 1/2 and add it to the second.
Also, multiply the third row by –1/2 and add it to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix A is $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

◆***Example 4*** Solve the following equation: $\frac{2}{3}x = 4$

Solution: To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$ which happens to be $\frac{3}{2}$. We get

$$\frac{3}{2} \cdot \frac{2}{3} x = 4 \cdot \frac{3}{2}$$

$$x = 6.$$

We use the Example 4 as an analogy to show how linear systems of the form AX = B are solved.

To solve a linear system, we first write the system in the matrix equation AX = B, where A is the coefficient matrix, X the matrix of variables, and B the matrix of constant terms.
We then multiply both sides of this equation by the multiplicative inverse of the matrix A.

Consider the following example.

◆***Example 5*** Solve the following system

$$3x + y = 3$$
$$5x + 2y = 4$$

Solution: To solve the above equation, first we express the system as

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation AX = B by A^{-1}.
Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example 2, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Multiplying both sides by A^{-1} , we get

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, x = 2, and y = –3.

◆***Example 6*** Solve the following system:

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solution: To solve the above equation, we write the system in matrix form AX = B as follows:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need inverse of A. From Example 3, $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

Multiplying both sides of the matrix equation AX = B on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix [A | I_n].
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is [I_n | B], then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

The Method for Solving a System of Equations When a Unique Solution Exists

1. Express the system in the matrix equation AX = B.
2. To solve the equation AX = B, we multiply on both sides by A^{-1}.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I\,X &= A^{-1}B \quad \text{where I is the identity matrix} \end{aligned}$$

SECTION 2.4 PROBLEM SET: INVERSE MATRICES

In problems 1- 2, verify that the given matrices are inverses of each other.

1) $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$	2) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 3 & -4 & 1 \\ 2 & -4 & 1 \\ 3 & -5 & 1 \end{bmatrix}$

In problems 3- 6, find the inverse of each matrix by the row-reduction method.

3) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$	4) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

SECTION 2.4 PROBLEM SET: INVERSE MATRICES

In problems 5- 6, find the inverse of each matrix by the row-reduction method.

5) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	6) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Problems 7-10: Express the system as AX = B; then solve using matrix inverses found in problems 3-6.

7) $\begin{aligned} 3x - 5y &= 2 \\ -x + 2y &= 0 \end{aligned}$	8) $\begin{aligned} x + \quad 2z &= 8 \\ y + 4z &= 8 \\ z &= 3 \end{aligned}$

SECTION 2.4 PROBLEM SET: INVERSE MATRICES

Problems 9-10: Express the system as AX = B; then solve using matrix inverses found in problems 3-6.

9) $\begin{aligned} x + y - z &= 2 \\ x + z &= 7 \\ 2x + y + z &= 13 \end{aligned}$	10) $\begin{aligned} x + y + z &= 2 \\ 3x + y &= 7 \\ x + y + 2z &= 3 \end{aligned}$
11) Why is it necessary that a matrix be a square matrix for its inverse to exist? Explain by relating the matrix to a system of equations.	12) Suppose we are solving a system AX = B by the matrix inverse method, but discover A has no inverse. How else can we solve this system? What can be said about the solutions of this system?

2.5 Application of Matrices in Cryptography

In this section you will learn to

1. encode a message using matrix multiplication.

2. decode a coded message using the matrix inverse and matrix multiplication

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indian, and Greek encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc, to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into to modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the 20th century. The military would encode messages before sending and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers, but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

◆***Example 1*** Use matrix A = $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ to encode the message: ATTACK NOW!

Solution: We divide the letters of the message into groups of two.

AT TA CK –N OW

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \qquad \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix} \qquad \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \end{bmatrix} \begin{bmatrix} 15 \\ 23 \end{bmatrix} \qquad \textbf{(I)}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A.

For example, the product of A with our first matrix is: $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$

And the product of A with our second matrix is: $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$

Multiplying each matrix in **(I)** by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 61 \end{bmatrix} \begin{bmatrix} 22 \\ 23 \end{bmatrix} \begin{bmatrix} 25 \\ 36 \end{bmatrix} \begin{bmatrix} 55 \\ 69 \end{bmatrix} \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

◆***Example 2*** Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix}\begin{bmatrix} 37 \\ 53 \end{bmatrix}\begin{bmatrix} 45 \\ 54 \end{bmatrix}\begin{bmatrix} 74 \\ 101 \end{bmatrix}\begin{bmatrix} 53 \\ 69 \end{bmatrix} \qquad \textbf{(II)}$$

Solution: Since this message was encoded by multiplying by the matrix A in Example 1, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example: $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$

By multiplying each of the matrices in **(II)** by the matrix A^{-1}, we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix}\begin{bmatrix} 5 \\ 16 \end{bmatrix}\begin{bmatrix} 27 \\ 9 \end{bmatrix}\begin{bmatrix} 20 \\ 27 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} K \\ E \end{bmatrix}\begin{bmatrix} E \\ P \end{bmatrix}\begin{bmatrix} - \\ I \end{bmatrix}\begin{bmatrix} T \\ - \end{bmatrix}\begin{bmatrix} U \\ P \end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a 3×3 matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

◆***Example 3*** Using the matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, encode the message: ATTACK NOW!

Solution: We divide the letters of the message into groups of three.

ATT ACK –NO W– –

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix} A \\ T \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \quad \begin{bmatrix} A \\ C \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \quad \begin{bmatrix} - \\ N \\ O \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \quad \begin{bmatrix} W \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}\begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix}\begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix} \qquad \textbf{(III)}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in **(III)** by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}\begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix}\begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix}\begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by B^{-1}, and associate the numbers with the corresponding letters of the alphabet.

In Example 4 we will demonstrate how to use matrix B^{-1} to decode an encrypted message.

◆***Example 4*** Decode the following message that was encoded using matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}\begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}\begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix} \qquad \textbf{(IV)}$$

Solution: Since this message was encoded by multiplying by the matrix B. We first determine inverse of B.

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in **(IV)** by the matrix B^{-1} gives the following.

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}\begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix}\begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix} \text{H} \\ \text{O} \\ \text{L} \end{bmatrix}\begin{bmatrix} \text{D} \\ - \\ \text{F} \end{bmatrix}\begin{bmatrix} \text{I} \\ \text{R} \\ \text{E} \end{bmatrix}$$ The message reads: HOLD FIRE.

We summarize:

TO ENCODE A MESSAGE

1. Divide the letters of the message into groups of two or three.
2. Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.
3. Convert each group of numbers into column matrices.
3. Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.

TO DECODE A MESSAGE

1. Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.
2. Associate the numbers with their corresponding letters.

SECTION 2.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 1- 8, the letters A to Z correspond to the numbers 1 to 26, as shown below, and a space is represented by the number 27.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

In problems 1 - 2, use the matrix A, given below, to encode the given messages.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

In problems 3 - 4, decode the messages that were encoded using matrix A.
Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

1) Encode the message: WATCH OUT!	2) Encode the message: HELP IS ON THE WAY.
3) Decode the following message: 64 23 102 41 82 32 97 35 71 28 69 32	4) Decode the following message: 105 40 117 48 39 19 69 32 72 27 37 15 114 47

SECTION 2.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 5 - 6, use the matrix B, given below, to encode the given messages.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

In problems 7 - 8, decode the messages that were encoded using matrix B.

Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

5) Encode the message using matrix B: LUCK IS ON YOUR SIDE.	6) Encode the message using matrix B: MAY THE FORCE BE WITH YOU.
7) Decode the following message that was encoded using matrix B: 8 23 7 4 47 – 2 15 102 –12 20 58 15 27 80 18 12 74 –7	8) Decode the following message that was encoded using matrix B: 12 69 – 3 11 53 9 5 46 –10 18 95 – 9 25 107 4 27 76 22 1 72 –26

2.6 Applications – Leontief Models

In this section we will examine an application of matrices to model economic systems.

In the 1930's, Wassily Leontief used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section we look at both the closed and the open models that he developed.

THE CLOSED MODEL

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

◆***Example 1*** We assume that in a village there is a farmer, carpenter, and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes 40% of the food he produces, and gives 40% to the carpenter, and 20% to the tailor. Thirty percent of the carpenter's production is consumed by himself, 40% by the farmer, and 30% by the carpenter. Fifty percent of the tailor's production is used by himself, 30% by the farmer, and 20% by the tailor. Write the matrix that describes this closed model.

Solution: The table below describes the above information.

	Proportion produced by the farmer	Proportion produced by the carpenter	Proportion produced by the tailor
The proportion used by the farmer	.40	.40	.30
The proportion used by the carpenter	.40	.30	.20
The proportion used by the tailor	.20	.30	.50

In a matrix form it can be written as follows.

$$A = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the **input-output matrix.** It is important that we read the matrix correctly. For example the entry A_{23}, the entry in row 2 and column 3, represents the following.

A_{23} = 20% of the tailor's production is used by the carpenter.

A_{33} = 50% of the tailor's production is used by the tailor.

◆***Example 2*** In Example 1 above, how much should each person get for his efforts?

Solution: We choose the following variables.

x = Farmer's pay   y = Carpenter's pay   z = Tailor's pay

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid x dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself .40x dollars, he pays .40y dollars to the carpenter, and .30z to the tailor. Since the expenses equal the wages, we get the following equation.

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$y = .40x + .30y + .20z$$

$$z = .20x + .30y + .50z$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as X = AX

Simplification results in the system of equations (I – A) X = 0

$$\begin{aligned} .60x - .40y - .30z &= 0 \\ -.40x + .70y - .20z &= 0 \\ -.20x - .30y + .50z &= 0 \end{aligned}$$

Solving for x, y, and z using the Gauss-Jordan method, we get

$$x = \frac{29}{26}\ t \qquad y = \frac{12}{13}\ t \qquad \text{and } z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let t = $2600, then we get

x = $2900   y = $2400   and z = $2600

Note: The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.

THE OPEN MODEL

The open model is more realistic, as it deals with the economy where sectors of the economy not only satisfy each other's' needs, but they also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same; that is, whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people, the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other 40 billion dollars worth of the food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and 50 billion dollars worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining 60 billion dollars worth by the consumer. We write the internal consumption in the following table, and express the demand as the matrix D.

	F produces	C produces	T produces
F uses	.20	.25	.10
C uses	.15	.10	.05
T uses	.10	.05	.15

The consumer demand for each industry in billions of dollars is given below.

$$D = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

◆***Example 3*** In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix D?

Solution: We choose the following variables.

x = Farmer's output

y = Carpenter's output

z = Tailor's output

In the closed model, our equation was X = AX, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be X = AX + D, where D represents the demand matrix.

We express it as follows:

$$X = AX + D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .20 & .25 & .10 \\ .15 & .10 & .05 \\ .10 & .05 & .15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

To solve this system, we write it as

$$X = AX + D$$

$$(I - A)X = D \qquad \text{where I is a 3 by 3 identity matrix}$$

$$X = (I - A)^{-1} D$$

$$I - A = \begin{bmatrix} .80 & -.25 & -.10 \\ -.15 & .90 & -.05 \\ -.10 & -.05 & .85 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

$$X = \begin{bmatrix} 83.7999 \\ 74.2341 \\ 84.8138 \end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

Farmer = \$83.7999   Carpenter = \$74.2341   Tailor = \$84.813

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix A.

◆***Example 4*** Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production ,in dollars.

	F	C	T	Demand	Total
F	40	50	60	100	250
C	30	40	40	110	220
T	20	30	30	120	200

The first row says that of the \$250 dollars worth of production by the industry F, \$40 is used by F, \$50 is used by C, \$60 is used by T, and the remainder of \$100 is used by the consumer. The other rows are described in a similar manner.

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix A.

Solution: We are being asked to determine the following:

How much of the production of each of the three industries, F, C, and T is required to produce one unit of F? In the same way, how much of the production of each of the three industries, F, C, and T is required to produce one unit of C? And finally, how much of the production of each of the three industries, F, C, and T is required to produce one unit of T?

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows:

To produce 250 units of F, we need to use 40 units of F, 30 units of C, and 20 units of T.

Therefore, to produce 1 unit of F, we need to use 40/250 units of F, 30/250 units of C, and 20/250 units of T.

To produce 220 units of C, we need to use 50 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of C, we need to use 50/220 units of F, 40/220 units of C, and 30/220 units of T.

To produce 200 units of T, we need to use 60 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of T, we need to use 60/200 units of F, 40/200 units of C, and 30/200 units of T.

We obtain the following matrix.

$$A = \begin{bmatrix} 40/250 & 50/220 & 60/200 \\ 30/250 & 40/220 & 40/200 \\ 20/250 & 30/220 & 30/200 \end{bmatrix} = \begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix}$$

Clearly AX + D = X

$$\begin{bmatrix} 40/250 & 50/220 & 60/200 \\ 30/250 & 40/220 & 40/200 \\ 20/250 & 30/220 & 30/200 \end{bmatrix}\begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix} + \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} = \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}$$

We summarize as follows:

LEONTIEF'S CLOSED MODEL

1. All consumption is within the industries. There is no external demand.
2. Input = Output
3. X = AX or (I – A)X = 0

LEONTIEF'S OPEN MODEL

1. In addition to internal consumption, there is an outside demand by the consumer.
2. Input = Output
3. X = AX + D or $X = (I - A)^{-1} D$

SECTION 2.6 PROBLEM SET: APPLICATIONS – LEONTIEF MODELS

1) Solve the following homogeneous system.

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y + z &= 0 \\ 4x + 3y + 2z &= 0 \end{aligned}$$

2) Solve the following homogeneous system.

$$\begin{aligned} x - y - z &= 0 \\ x - 3y + 2z &= 0 \\ 2x - 4y + z &= 0 \end{aligned}$$

3) Chris and Ed decide to help each other by doing repairs on each others houses. Chris is a carpenter, and Ed is an electrician. Chris does carpentry work on his house as well as on Ed's house. Similarly, Ed does electrical repairs on his house and on Chris' house. When they are all finished they realize that Chris spent 60% of his time on his own house, and 40% of his time on Ed's house. On the other hand Ed spent half of his time on his house and half on Chris's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

4) Chris, Ed, and Paul decide to help each other by doing repairs on each others houses. Chris is a carpenter, Ed is an electrician, and Paul is a plumber. Each does work on his own house as well as on the others houses. When they are all finished they realize that Chris spent 30% of his time on his own house, 40% of his time on Ed's house, and 30% on Paul's house. Ed spent half of his time on his own house, 30% on Chris' house, and remaining on Paul's house. Paul spent 40% of the time on his own house, 40% on Chris' house, and 20% on Ed's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

SECTION 2.6 PROBLEM SET: APPLICATIONS – LEONTIEF MODELS

5) Given the internal consumption matrix A, and the external demand matrix D as follows.

$$A = \begin{bmatrix} .30 & .20 & .10 \\ .20 & .10 & .30 \\ .10 & .20 & .30 \end{bmatrix} \qquad D = \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}$$

Solve the system using the open model: $X = AX + D$ or $X = (I - A)^{-1}D$

6) Given the internal consumption matrix A, and the external demand matrix D as follows.

$$A = \begin{bmatrix} .05 & .10 & .10 \\ .10 & .15 & .05 \\ .05 & .20 & .20 \end{bmatrix} \qquad D = \begin{bmatrix} 50 \\ 100 \\ 80 \end{bmatrix}$$

Solve the system using the open model: $X = AX + D$ or $X = (I - A)^{-1}D$

7) An economy has two industries, farming and building. For every $1 of food produced, the farmer uses $.20 and the builder uses $.15. For every $1 worth of building, the builder uses $.25 and the farmer uses $.20. If the external demand for food is $100,000, and for building $200,000, what should be the total production for each industry in dollars?

SECTION 2.6 PROBLEM SET: APPLICATIONS – LEONTIEF MODELS

8) An economy has three industries, farming, building, and clothing. For every $1 of food produced, the farmer uses $.20, the builder uses $.15, and the tailor $.05. For every $1 worth of building, the builder uses $.25, the farmer uses $.20, and the tailor $.10. For every $1 worth of clothing, the tailor uses $.10, the builder uses $.20, the farmer uses $.15. If the external demand for food is $100 million, for building $200 million, and for clothing $300 million, what should be the total production for each in dollars?

9) Suppose an economy consists of three industries F, C, and T. The following table gives information about the internal use of each industry's production and external demand in dollars.

	F	C	T	Demand	Total
F	30	10	20	40	100
C	20	30	20	50	120
T	10	10	30	60	110

Find the proportion of the amounts consumed by each of the industries; that is, find the matrix A.

10) If in problem 9, the consumer demand for F, C, and T becomes 60, 80, and 100, respectively, find the total output and the internal use by each industry to meet that demand.

SECTION 2.7 PROBLEM SET: CHAPTER REVIEW

1) To reinforce her diet, Mrs. Tam bought a bottle containing 30 tablets of Supplement A and a bottle containing 50 tablets of Supplement B. Each tablet of supplement A contains 1000 mg of calcium, 400 mg of magnesium, and 15 mg of zinc, and each tablet of supplement B contains 800 mg of calcium, 500 mg of magnesium, and 20 mg of zinc.

   a) Represent the amount of calcium, magnesium and zinc in each tablet as a 2×3 matrix.

   b) Represent the number of tablets in each bottle as a row matrix.

   c) Use matrix multiplication to find the total amount of calcium, magnesium, and zinc in both bottles.

2) Let matrix $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 & -1 \\ 1 & 4 & -3 \end{bmatrix}$. Find the following.

   a) $\frac{1}{2}(A + B)$ b) $3A - 2B$

3) Let matrix $C = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & -3 & -1 \\ 3 & -1 & -2 \\ 3 & -3 & -2 \end{bmatrix}$. Find the following.

   a) $2(C - D)$ b) $C - 3D$

4) Let matrix $E = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -3 \end{bmatrix}$. Find the following.

   a) 2EF   b) 3FE

5) Let matrix $G = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Find the following.

   a) 2GH   b) HG

6) Solve the following systems using the Gauss-Jordan Method.

   a) $\begin{aligned} x + 3y - 2z &= 7 \\ 2x + 7y - 5z &= 16 \\ x + 5y - 3z &= 10 \end{aligned}$ b) $\begin{aligned} 2x - 4y + 4z &= 2 \\ 2x + y + 9z &= 17 \\ 3x - 2y + 2z &= 7 \end{aligned}$

7) An apple, a banana and three oranges or two apples, two bananas, and an orange, or four bananas and two oranges cost $2. Find the price of each.

8) Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then determine one particular solution.

   a) $\begin{aligned} x + y + z &= 6 \\ 2x - 3y + 2z &= 12 \\ 3x - 2y + 3z &= 18 \end{aligned}$ b) $\begin{aligned} x + y + 3z &= 4 \\ x + z &= 1 \\ 2x - y &= 2 \end{aligned}$

9) Elise has a collection of 12 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all possible solutions.

SECTION 2.7 PROBLEM SET: CHAPTER REVIEW

10) Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then find a particular solution.

a) $$\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= 8 \\ 3x + 6y - 3z &= 3 \end{aligned}$$

b) $$\begin{aligned} x - 2y + 2z &= 1 \\ 2x - 3y + 5z &= 4 \end{aligned}$$

11) Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then provide one particular solution.

a) $$\begin{aligned} 2x + y - 2z &= 0 \\ 2x + 2y - 3z &= 0 \\ 6x + 4y - 7z &= 0 \end{aligned}$$

b) $$\begin{aligned} 3x + 4y - 3z &= 5 \\ 2x + 3y - z &= 4 \\ x + 2y + z &= 1 \end{aligned}$$

12) Find the inverse of the following matrices: a) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

13) Solve the following systems using the matrix inverse method.

a) $$\begin{aligned} 2x + 3y + z &= 12 \\ x + 2y + z &= 9 \\ x + y + z &= 5 \end{aligned}$$

b) $$\begin{aligned} x + 2y - 3z + w &= 7 \\ x - z &= 4 \\ x - 2y + z &= 0 \\ y - 2z + w &= -1 \end{aligned}$$

14) Use matrix A to encode the following messages.
The space between the letters is represented by the number 27, and all punctuation is ignored.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

a) TAKE IT AND RUN. b) GET OUT QUICK.

15) Decode the following messages that were encoded using matrix A in the above problem.

a) 44, 71, 15, 18, 27, 1, 68, 82, 27, 69, 76, 27, 19, 33, 9

b) 37, 64, 15, 36, 54, 15, 67, 75, 20, 59, 66, 27, 39, 43, 12

16) Chris, Bob, and Matt decide to help each other study during the final exams. Chris's favorite subject is chemistry, Bob loves biology, and Matt knows his math. Each studies his own subject as well as helps the others learn their subjects. After the finals, they realize that Chris spent 40% of his time studying his own subject chemistry, 30% of his time helping Bob learn chemistry, and 30% of the time helping Matt learn chemistry. Bob spent 30% of his time studying his own subject biology, 30% of his time helping Chris learn biology, and 40% of the time helping Matt learn biology. Matt spent 20% of his time studying his own subject math, 40% of his time helping Chris learn math, and 40% of the time helping Bob learn math. If they originally agreed that each should work about 33 hours, how long did each work?

17) As in the previous problem, Chris, Bob, and Matt decide to not only help each other study during the final exams, but also tutor others to make a little money. Chris spends 30% of his time studying chemistry, 15% of his time helping Bob with chemistry, and 25% helping Matt with chemistry. Bob spends 25% of his time studying biology, 15% helping Chris with biology, and 30% helping Matt. Similarly, Matt spends 20% of his time on his own math, 20% helping Chris, and 20% helping Bob. If they spend respectively, 12, 12, and 10 hours tutoring others, how many total hours are they going to end up working?

CHAPTER 3

Chapter 3: Linear Programming: A Geometrical Approach

In this chapter, you will learn to:

1. *Solve linear programming problems that maximize the objective function.*
2. *Solve linear programming problems that minimize the objective function.*

3.1 Maximization Applications

In this section, you will learn to:

1. *recognize the typical form of a linear programing problem*
2. *Formulate maximization linear programming problems*
3. *Graph feasibility regions for maximization linear programming problems*
4. *Determine optimal solutions for maximization linear programming problems.*

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical **linear programming** problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The function we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing.

In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

◆***Example 1*** Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.
If Nikki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution: We start by choosing our variables.

Let x = The number of hours per week Niki will work at Job I.

y = The number of hours per week Niki will work at Job II.

Now we write the objective function. Since Niki gets paid $40 an hour at Job I, and $30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

$$x + y \leq 12$$

The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." The translation follows.

$$2x + y \leq 16$$

The fact that x and y can never be negative is represented by the following two constraints:

$x \geq 0$, and $y \geq 0$.

Well, good news! We have formulated the problem. We restate it as

Maximize $I = 40x + 30y$

Subject to: $x + y \leq 12$

$2x + y \leq 16$

$x \geq 0;\ y \geq 0$

In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints.

Any appropriate method can be used to graph the lines for the constraints. However often the easiest method is to graph the line by plotting the x-intercept and y-intercept.

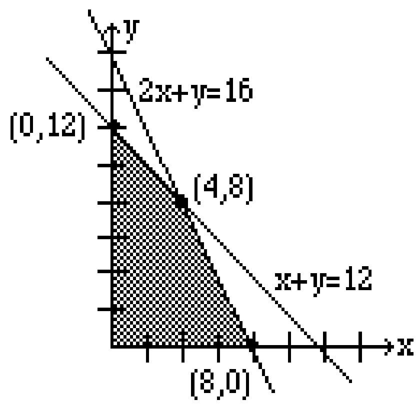
The line for a constraint will divide the plane into two region, one of which satisfies the inequality part of the constraint. A test point is used to determine which portion of the plane to shade to satisfy the inequality. Any point on the plane that is not on the line can be used as a test point.

- If the test point satisfies the inequality, then the region of the plane that satisfies the inequality is the region that contains the test point.
- If the test point does not satisfy the inequality, then the region that satisfies the inequality lies on the opposite side of the line from the test point.

In the graph below, after the lines representing the constraints were graphed using an appropriate method from Chapter 1, the point (0,0) was used as a test point to determine that

- (0,0) satisfies the constraint $x + y \leq 12$ because $0 + 0 < 12$
- (0,0) satisfies the constraint $2x + y \leq 16$ because $2(0) + 0 < 16$

Therefore, in this example, we shade the region that is below and to the left of **both** constraint lines, but also above the x axis and to the right of the y axis, in order to further satisfy the constraints $x \geq 0$ and $y \geq 0$.



The shaded region where all conditions are satisfied is called the **feasibility region** or the feasibility polygon.

The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region.

Therefore, we will identify all the vertices (corner points) of the feasibility region. We call these points **critical points.** They are listed as (0, 0), (0, 12), (4, 8), (8, 0). To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below.

Critical points	Income	
(0, 0)	40(0) + 30(0)	= $0
(0, 12)	40(0) + 30(12)	= $360
(4, 8)	40(4) + 30(8)	= $400
(8,0)	40(8) + 30(0)	= $320

Clearly, the point (4, 8) gives the most profit: $400.

Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

◆***Example 2*** A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of \$20 is realized for each regular gadget and \$30 for a premium gadget, how many of each should be manufactured to maximize profit?

Solution: We choose our variables.

Let x = The number of regular gadgets manufactured each day.

and y = The number of premium gadgets manufactured each day.

The objective function is

$$P = 20x + 30y$$

We now write the constraints. The fourth sentence states that the company can make at most 7 gadgets a day. This translates as

$$x + y \leq 7$$

Since the regular gadget requires one hour of assembly and the premium gadget requires two hours of assembly, and there are at most 12 hours available for this operation, we get

$$x + 2y \leq 12$$

Similarly, the regular gadget requires two hours of finishing and the premium gadget one hour. Again, there are at most 12 hours available for finishing. This gives us the following constraint.

$$2x + y \leq 12$$

The fact that x and y can never be negative is represented by the following two constraints:

$x \geq 0$, and $y \geq 0$.

We have formulated the problem as follows:

Maximize $P = 20x + 30y$

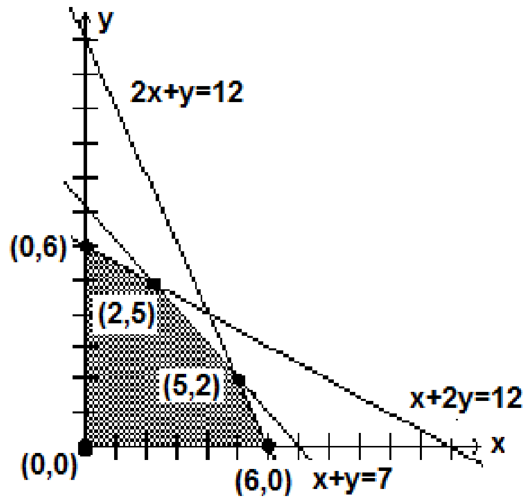
Subject to: $x + y \leq 7$

$x + 2y \leq 12$

$2x + y \leq 12$

$x \geq 0$; $y \geq 0$

In order to solve the problem, we next graph the constraints and feasibility region.



Again, we have shaded the feasibility region, where all constraints are satisfied.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0). To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below.

Critical point	Income	
(0, 0)	20(0) + 30(0)	= $0
(0, 6)	20(0) + 30(6)	= $180
(2, 5)	20(2) + 30(5)	= $190
(5, 2)	20(5) + 30(2)	= $160
(6,0)	20(6) + 30(0)	= $120

The point (2, 5) gives the most profit, and that profit is $190.
Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of $190.

So far we have focused on "**standard maximization problems**" in which

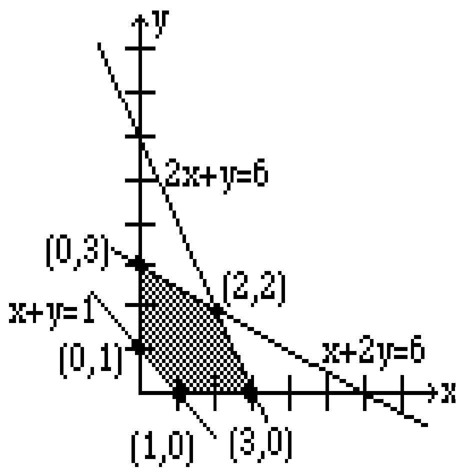
1. The objective function is to be maximized
2. All constraints are of the form $ax + by \le c$
3. All variables are constrained to by non-negative ($x \ge 0$, $y \ge 0$)

We will next consider an example where that is not the case. Our next problem is said to have "**mixed constraints**", since some of the inequality constraints are of the form $ax + by \le c$ and some are of the form $ax + by \ge c$. The non-negativity constraints are still an important requirement in any linear program.

◆***Example 3*** Solve the following maximization problem graphically.

Maximize	$P = 10x + 15y$
Subject to:	$x + y \geq 1$
	$x + 2y \leq 6$
	$2x + y \leq 6$
	$x \geq 0;\ y \geq 0$

Solution: The graph is shown below.



The five critical points are listed in the above figure. The reader should observe that the first constraint $x + y \geq 1$ requires that the feasibility region must be bounded below by the line $x + y = 1$; the test point (0,0) does not satisfy $x + y \geq 1$, so we shade the region on the opposite side of the line from test point (0,0).

Critical point	Income	
(1, 0)	10(1) + 15(0)	= $10
(3, 0)	10(3) + 15(0)	= $30
(2, 2)	10(2) + 15(2)	= $50
(0, 3)	10(0) + 15(3)	= $45
(0,1)	10(0) + 15(1)	= $15

Clearly, the point (2, 2) maximizes the objective function to a maximum value of 50.

It is important to observe that that if the point (0,0) lies on the line for a constraint, then (0,0) could not be used as a test point. We would need to select any other point we want that does not lie on the line to use as a test point in that situation.

Finally, we address an important question. Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

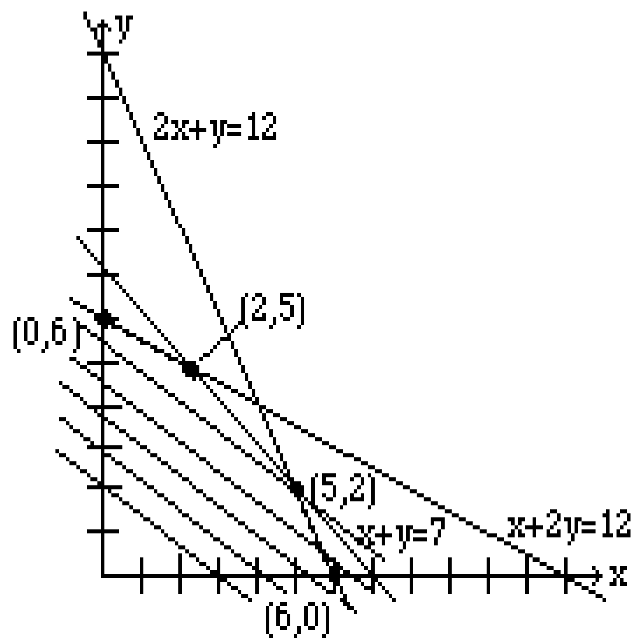
For example 2, we substituted the points (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0), in the objective function P = 20x + 30y, and we got the values $0, $180, $190, $160, $120, respectively.

Sometimes that is not the most efficient way of finding the optimum solution. Instead we could find the optimal value by also graphing the objective function.

To determine the largest P, we graph P = 20x + 30y for any value P of our choice. Let us say, we choose P = 60. We graph 20x + 30y = 60.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point (2, 5). In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.



We summarize:

The Maximization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
   a) For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$
   b) Since the variables are non-negative, we include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the maximum value.
   a) This is done by finding the value of the objective function at each corner point.
   b) This can also be done by moving the line associated with the objective function.

SECTION 3.1 PROBLEM SET: MAXIMIZATION APPLICATIONS

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

2) Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

SECTION 3.1 PROBLEM SET: MAXIMIZATION APPLICATIONS

3) A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize revenue?

4) The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?
(Hint: Use consistent units. Work the entire problem in pounds by converting all values given in ounces into fractions of pounds).

SECTION 3.1 PROBLEM SET: MAXIMIZATION APPLICATIONS

5) Maximize: $Z = 4x + 10y$

Subject to:
$x + y \leq 5$
$2x+y \leq 8$
$x+2y \leq 8$
$x \geq 0, y \geq 0$

6) This maximization linear programming problem is not in "standard" form. It has mixed constraints, some involving $\leq$ inequalities and some involving $\geq$ inequalities. However with careful graphing, we can solve this using the techniques we have learned in this section.

Maximize $Z = 5x + 7y$

Subject to
$x + y \leq 30$
$2x+ y \leq 50$
$4x+3y \geq 60$
$2x \geq y$
$x \geq 0, y \geq 0$

3.2 Minimization Applications

In this section, you will learn to:

1. *Formulate minimization linear programming problems*
2. *Graph feasibility regions for maximization linear programming problems*
3. *Determine optimal solutions for maximization linear programming problems.*

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the **standard minimization linear program**, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

◆***Example 1*** At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns $15 per hour for grading papers. Mary is an post-doctoral associate and can grade 30 papers per hour; Mary earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment.
If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Solution: We choose the variables as follows:

Let x = The number of hours per week John is employed.

and y = The number of hours per week Mary is employed.

The objective function is

$$C = 15x + 25y$$

The fact that each must work at least one hour each week results in the following two constraints:

$$x \geq 1$$
$$y \geq 1$$

Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110$$

The fact that x and y are non-negative, we get

$$x \geq 0, \text{ and } y \geq 0.$$

The problem has been formulated as follows.

Minimize $C = 15x + 25y$

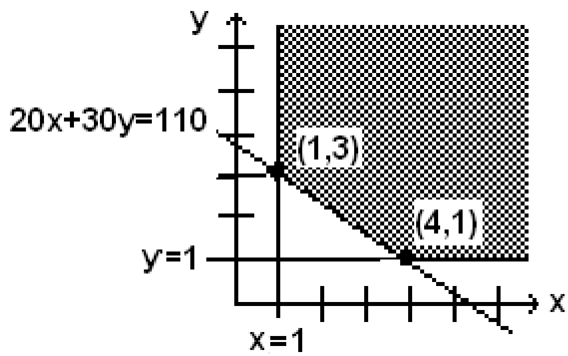
Subject to: $x \geq 1$

$y \geq 1$

$20x + 30y \geq 110$

$x \geq 0$; $y \geq 0$

To solve the problem, we graph the constraints as follows:



Again, we have shaded the feasibility region, where all constraints are satisfied.

If we used test point (0,0) that does not lie on any of the constraints, we observe that (0, 0) **does not** satisfy any of the constraints $x \geq 1$, $y \geq 1$, $20x + 30y \geq 110$. Thus all the shading for the feasibility region lies on the opposite side of the constraint lines from the point (0,0).

Alternatively we could use test point (4,6), which also does not lie on any of the constraint lines. We'd find that (4,6) **does** satisfy all of the inequality constraints. Consequently all the shading for the feasibility region lies on the same side of the constraint lines as the point (4,6).

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, (1, 3) and (4, 1). To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

Critical points	Income
(1, 3)	15(1) + 25(3) = \$90
(4, 1)	15(4) + 25(1) = \$85

The point (4, 1) gives the least cost, and that cost is \$85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of \$85 per week.

◆***Example 2*** Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

	PASTA	**TOFU**
PROTEIN	8g	16g
CARBOHYDRATES	60g	40g
VITAMIN C	2g	2g
CHOLESTEROL	60mg	50mg

Solution: We choose the variables as follows.

Let x = The number of days Mr. Hamer eats Pasta.

and y = The number of days Mr. Hamer eats Tofu.

Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y$$

The constraint associated with the total amount of protein provided by both meals is

$$8x + 16y \geq 200$$

Similarly, the two constraints associated with the total amount of carbohydrates and vitamins are obtained, and they are

$$60x + 40y \geq 960$$

$$2x + 2y \geq 40$$

The constraints that state that x and y are non-negative are

$x \geq 0$, and $y \geq 0$.

We summarize all information as follows:

Minimize $C = 60x + 50y$

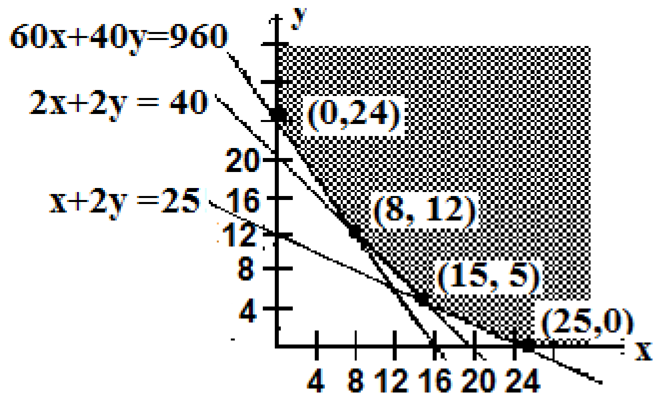
Subject to: $8x + 16y \geq 200$

$60x + 40y \geq 960$

$2x + 2y \geq 40$

$x \geq 0$; $y \geq 0$

To solve the problem, we graph the constraints and shade the feasibility region.



We have shaded the unbounded feasibility region, where all constraints are satisfied.

To minimize the objective function, we find the vertices of the feasibility region. These vertices are (0, 24), (8, 12), (15, 5) and (25, 0). To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

Critical points	Income
(0, 24)	60(0) + 50(24) = 1200
(8, 12)	60(8) + 50(12) = 1080
(15, 5)	60(15) + 50(5) = 1150
(25, 0)	60(25) + 50(0) = 1500

The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta 8 days, and Tofu 12 days.

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution is if there is not a feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies **all** the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it can not have an optimal solution.
- A linear program can fail to have an optimal solution if the feasibility region is unbounded.

  The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions.

  However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find sother values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

Minimization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
   a) For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b) Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the minimum value.
   a) This can be done by finding the value of the objective function at each corner point.
   b) This can also be done by moving the line associated with the objective function.
   c) There is the possibility that the problem has no solution.

SECTION 3.2 PROBLEM SET: MINIMIZATION APPLICATIONS

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs $2 per unit and Food B cost $1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a $60 commission for each laptop, and a $40 commission for each desktop. Let x = the number of laptops and y = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people? What is the minimum commission?

SECTION 3.2 PROBLEM SET: MINIMIZATION APPLICATIONS

3) An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil. How many days should each refinery be operated to meet the goals at a minimum cost?

4) A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of $800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of $1000 per month. How many months should each of the two contractors be employed to minimize the cost?

SECTION 3.3 PROBLEM SET: CHAPTER REVIEW

Solve the following linear programming problems by the graphical method.

1) Mr. Shoemacher has $20,000 to invest in two types of mutual funds: a High-Yield Fund and an Equity Fund. The High-Yield fund has an annual yield of 12%, while the Equity fund earns 8%. He would like to invest at least $3000 in the High-Yield fund and at least $4000 in the Equity fund. How much should he invest in each to maximize his annual yield, and what is the maximum yield?

2) Dr. Lum teaches part-time at two community colleges, Hilltop College and Serra College. Dr. Lum can teach up to 5 classes per semester. For every class he teaches at Hilltop College, he needs to spend 3 hours per week preparing lessons and grading papers. For each class at Serra College, he must do 4 hours of work per week. He has determined that he cannot spend more than 18 hours per week preparing lessons and grading papers. If he earns $6,000 per class at Hilltop College and $7,500 per class at Serra College, how many classes should he teach at each college to maximize his income, and what will be his income?

3) Mr. Shamir employs two part-time typists, Inna and Jim, for his typing needs. Inna charges $15 an hour and can type 6 pages an hour, while Jim charges $18 an hour and can type 8 pages per hour. Each typist must be employed at least 8 hours per week to keep them on the payroll. If Mr. Shamir has at least 208 pages to be typed, how many hours per week should he employ each typist to minimize his typing costs, and what will be the total cost?

4) Mr. Boutros wants to invest up to $20,000 in two stocks, Cal Computers and Texas Tools. The Cal Computers stock is expected to yield a 16% annual return, while the Texas Tools stock promises a 12% yield. Mr. Boutros would like to earn at least $2,880 this year. According to Value Line Magazine's safety index (1 highest to 5 lowest), Cal Computers has a safety number of 3 and Texas Tools has a safety number of 2. How much money should he invest in each to minimize the safety number? Note: A lower safety number means less risk.

5) A store sells two types of copy machines: compact (low capacity) and standard (which takes more space). The store can sell up to 90 copiers a month. A maximum of 1080 cubic feet of storage space is available. A compact copier requires 6 cu. ft. of storage space, and a standard copier requires 18 cu. ft.. The compact and standard copy machines take, respectively, 1 and 1.5 sales hours of labor.
A maximum of 99 hours of labor is available. The profit from each of these copiers is $60 and $80, respectively, how many of each type should be sold to maximize profit, and what is the maximum profit?

6) A company manufactures two types of cell phones, a Basic model and a Pro model. The Basic model generates a profit of $100 per phone and the Pro model has a profit of $150 per phone. On the assembly line the Basic phone requires 7 hours, while the Pro model takes 11 hours. The Basic phone requires one hour and the Pro phone needs 3 hours for finishing, which includes loading software. Both phones require one hour for testing. On a particular production run the company has available 1,540 work hours on the assembly line, 360 work hours for finishing, and 200 work hours in the testing department. How many cell phones of each type should be produced to maximize profit, and what is that maximum profit?

7) John wishes to choose a combination of two types of cereals for breakfast - Cereal A and Cereal B.
A small box (one serving) of Cereal A costs $0.50 and contains 10 units of vitamins, 5 units of minerals, and 15 calories. A small box(one serving) of Cereal B costs $0.40 and contains 5 units of vitamins, 10 units of minerals, and 15 calories. John wants to buy enough boxes to have at least 500 units of vitamins, 600 units of minerals, and 1200 calories. How many boxes of each food should he buy to minimize his cost, and what is the minimum cost?

SECTION 3.3 PROBLEM SET: CHAPTER REVIEW

8) Jessica needs at least 60 units of vitamin A, 40 units of vitamin B, and 140 units of vitamin C each week. She can choose between Costless brand or Savemore brand tablets. A Costless tablet costs 5 cents and contains 3 units of vitamin A, 1 unit of vitamin B, and 2 units of vitamin C. A Savemore tablet costs 7 cents and contains 1 unit of A, 1 of B, and 5 of C. How many tablets of each kind should she buy to minimize cost, and what is the minimum cost?

9) A small company manufactures two products: A and B. Each product requires three operations: Assembly, Finishing and Testing. Product A requires 1 hour of Assembly, 3 hours of Finishing, and 1 hour of Testing. Product B requires 3 hours of Assembly, 1 hour of Finishing, and 1 hour of Testing. The total work-hours available per week in the Assembly division is 60, in Finishing is 60, and in Testing is 24. Each item of product A has a profit of $50, and each item of Product B has a profit of $75. How many of each should be made to maximize profit? What is the maximum profit?

10) A factory manufactures two products, A and B. Each product requires the use of three machines, Machine I, Machine II, and Machine III. The time requirements and total hours available on each machine are listed below.

	Machine I	Machine II	Machine III
Product A	1	2	4
Product B	2	2	2
Total hours	70	90	160

If product A generates a profit of $60 per unit and product B a profit of $50 per unit, how many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three types of shoes, formal, casual, and athletic, at its two factories, Factory I and Factory II. The company must produce at least 6000 pairs of formal shoes, 8000 pairs of casual shoes, and 9000 pairs of athletic shoes. Daily production of each factory for each type of shoe is:

	Factory I	Factory II
Formal	100	100
Casual	100	200
Athletic	300	100

Operating Factory I costs $1500 per day and it costs $2000 per day to operate Factory II. How many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) A professor gives two types of quizzes, objective and recall. He plans to give at least 15 quizzes this quarter. The student preparation time for an objective quiz is 15 minutes and for a recall quiz 30 minutes. The professor would like a student to spend at least 5 hours (300 minutes) preparing for these quizzes above and beyond the normal study time. The average score on an objective quiz is 7, and on a recall type 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type should he give in order to minimize his grading time?

13) A company makes two mixtures of nuts: Mixture A and Mixture B. Mixture A contains 30% peanuts, 30% almonds and 40% cashews and sells for $5 per pound. Mixture B contains 30% peanuts, 60% almonds and 10% cashews and sells for $3 a pound. The company has 540 pounds of peanuts, 900 pounds of almonds, 480 pounds of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit, and what is the maximum profit?

CHAPTER 4

Chapter 4: Linear Programming: The Simplex Method

In this chapter, you will:

1. *Investigate real world applications of linear programming and related methods.*
2. *Solve linear programming maximization problems using the simplex method.*
3. *Solve linear programming minimization problems using the simplex method.*

4.1 Linear Programming Applications in Business, Finance, Medicine, and Social Science

In this section, you will learn about real world applications of linear programming and related methods.

The linear programs we solved in chapter 3 contain only two variables, x and y, so that we could solve them graphically. In practice, linear programs can contain thousands of variables and constraints.

Later in this chapter we'll learn to solve linear programs with more than two variables using the simplex algorithm, which is a numerical solution method that uses matrices and row operations. However, in order to make the problems practical for learning purposes, our problems will still have only several variables.

Now that we understand the main concepts behind linear programming, we can also consider how linear programming is currently used in large scale real-world applications.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems where solutions must be integers are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering hand management science are among the terms used to describe mathematical modelling techniques that may include linear programming and related met

In the rest of this section we'll explore six real world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

AIRLINE SCHEDULING

Airlines use techniques that include and are related to linear programming to schedule their aircrafts to flights on various routes, and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs. Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.
- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for number of passengers.
- Aircraft must be compatible with the airports it departs from and arrives at – not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case the considerations to be managed involve:

- Getting aircrafts and crews back on schedule as quickly as possible
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

KIDNEY DONATION CHAIN:

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donor allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.
- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors. The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

ADVERTISEMENTS IN ONLINE MARKETING

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased?

Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual's previous browsing and purchase selections, he or she is assigned a "propensity score" for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual's history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company's goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modelled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

LOANS

A car manufacturer sells its cars though dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a future chapter we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer's credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise the customer may find another lender, such as a bank, which can offer a more favorable loan.

PRODUCTION PLANNING AND SCHEDULING IN MANUFACTURING

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make.

There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers.

The linear program that monitors production planning and scheduling must be updated frequently – daily or even twice each day – to take into account variations from a master plan.

BIKE SHARE PROGRAMS

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 9, we'll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the desire distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

4.2 Maximization by the Simplex Method

In this section, you will learn to solve linear programming maximization problems using the Simplex Method:

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau.

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **simplex method**.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but as it turned out it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method.

But first, we list the algorithm for the simplex method.

THE SIMPLEX METHOD

1. **Set up the problem.**
   That is, write the objective function and the inequality constraints.

2. **Convert the inequalities into equations.**
   This is done by adding one slack variable for each inequality.

3. **Construct the initial simplex tableau.**
   Write the objective function as the bottom row.

4. **The most negative entry in the bottom row identifies the pivot column.**

5. **Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.**
   The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.

6. **Perform pivoting to make all other entries in this column zero.**
   This is done the same way as we did with the Gauss-Jordan method.

7. **When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.**

8. **Read off your answers.**

   Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

Now, we use the simplex method to solve Example 1 solved geometrically in section 3.1.

◆***Example 1*** Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution: In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.

Let x_1 = The number of hours per week Niki will work at Job I.

and x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated the same way as we did in the last chapter.

Maximize $Z = 40x_1 + 30x_2$

Subject to:

$$x_1 + x_2 \le 12$$
$$2x_1 + x_2 \le 16$$
$$x_1 \ge 0; \; x_2 \ge 0$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1 + x_2 \le 12$ into an equation, we add a non-negative variable y_1, and we get

$$x_1 + x_2 + y_1 = 12$$

Here the variable y_1 picks up the slack, and it represents the amount by which $x_1 + x_2$ falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + Z = 0$.

After adding the slack variables, our problem reads

Objective function: $-40x_1 - 30x_2 + Z = 0$

Subject to constraints:

$$x_1 + x_2 + y_1 = 12$$
$$2x_1 + x_2 + y_2 = 16$$
$$x_1 \ge 0; \; x_2 \ge 0$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\begin{bmatrix} y_1 & y_2 & Z & | & C \\ 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

which reads

$y_1 = 12$ $\quad$ $y_2 = 16$ $\quad$ $Z = 0$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is –40; therefore the column 1 is identified.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z
↑						

◆***Question*** Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function – the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $Z = 40x_1 + 30x_2$, it will make more sense to increase the value of x_1 rather than x_2. The variable x_1 represents the number of hours per week Niki works at Job I. Since Job I pays $40 per hour as opposed to Job II which pays only $30, the variable x_1 will increase the objective function by $40 for a unit of increase in the variable x_1.

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x_1	x_2	y_1	y_2	Z			
1	1	1	0	0	12	y_1	$12 \div 1 = 12$
[2]	1	0	1	0	16	y_2	$\leftarrow 16 \div 2 = 8$
–40 ↑	–30	0	0	1	0	Z	

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

◆***Question*** Why do we find quotients, and why does the smallest quotient identify a row?

Answer When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1. But we cannot choose any value for x_1. Can we let $x_1 = 100$? Definitely not! That is because Niki never wants to work for more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Can we let $x_1 = 12$? Again, the answer is no because the preparation time for Job I is two times the time spent on the job. Since Niki never wants to spend more than 16 hours for preparation, the maximum time she can work is $16 \div 2 = 8$.

Now you see the purpose of computing the quotients; using the quotients to identify the pivot element guarantees that we do not violate the constraints.

◆***Question*** Why do we identify the pivot element?

Answer As we have mentioned earlier, the simplex method begins with a corner point and then moves to the next corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the **entering variable,** and the variable whose units are being replaced is called the **departing variable**. The entering variable in the above table is x_1, and it was identified by the most negative entry in the bottom row. The departing variable y_2 was identified by the lowest of all quotients.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. So now our job is to make our pivot element a 1 by dividing the entire second row by 2. The result follows.

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by –1 and add it to row 1. We get

x_1	x_2	y_1	y_2	Z	
0	1/2	1	–1/2	0	4
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x_1	x_2	y_1	y_2	Z		
0	1/2	1	–1/2	0	4	y_1
[1]	1/2	0	1/2	0	8	x_1
0	–10	0	20	1	320	Z

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $y_2 = 0$, we obtain $x_1 = 8$, $y_1 = 4$, and z = 320. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\begin{bmatrix} x_1 & y_1 & Z & | & C \\ 0 & 1 & 0 & | & 4 \\ 1 & 0 & 0 & | & 8 \\ 0 & 0 & 1 & | & 320 \end{bmatrix}$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $y_1 = 4$, $y_2 = 0$ and z = 320. At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be $320. Recall from Example 1 in section 3.1 that (8, 0) was one of our corner points. Here $y_1 = 4$ and $y_2 = 0$ mean that she will be left with 4 hours of working time and no preparation time.

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, –10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x_1	x_2	y_1	y_2	Z			
0	[1/2]	1	–1/2	0	4	y_1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x_1	8 ÷ 1/2 = 16
0	–10	0	20	1	320	Z	
	↑						

We make the pivot element 1 by multiplying row 1 by 2, and we get

x_1	x_2	y_1	y_2	Z	
0	[1]	2	–1	0	8
1	1/2	0	1/2	0	8
0	–10	0	20	1	320

Now to make all other entries as zeros in this column, we first multiply row 1 by –1/2 and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.

x_1	x_2	y_1	y_2	Z		
0	1	2	–1	0	8	x_2
1	0	–1	1	0	4	x_1
0	0	20	10	1	400	Z

We no longer have negative entries in the bottom row, therefore we are finished.

◆***Question*** Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \text{or}$$

$$Z = 400 - 20y_1 - 10y_2$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when y_1 and y_2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labeled y_1 and y_2 are not such columns, we arbitrarily choose $y_1 = 0$, and $y_2 = 0$, and we get

x_1	x_2	Z	C
0	1	0	8
1	0	0	4
0	0	1	400

The matrix reads $x_1 = 4$, $x_2 = 8$ and $z = 400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to $400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

SECTION 4.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

1) Maximize $z = x_1 + 2x_2 + 3x_3$

subject to

$$x_1 + x_2 + x_3 \leq 12$$
$$2x_1 + x_2 + 3x_3 \leq 18$$
$$x_1, x_2, x_3 \geq 0$$

---

2) Maximize $z = x_1 + 2x_2 + x_3$

subject to

$$x_1 + x_2 \leq 3$$
$$x_2 + x_3 \leq 4$$
$$x_1 + x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

---

3) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

SECTION 4.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

4) A factory manufactures chairs, tables and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair, $30 for a table, and $25 for a bookcase, how many units of each should be manufactured to maximize profit?

---

5). The Acme Apple company sells its Pippin, Macintosh, and Fuji apples in mixes. Box I contains 4 apples of each kind; Box II contains 6 Pippin, 3 Macintosh, and 3 Fuji; and Box III contains no Pippin, 8 Macintosh and 4 Fuji apples. At the end of the season, the company has altogether 2800 Pippin, 2200 Macintosh, and 2300 Fuji apples left. Determine the maximum number of boxes that the company can make.

4.3 Minimization by the Simplex Method

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form

2. Formulate a dual problem in standard maximization form

3. Use the simplex method to solve the dual maximization problem

4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the **dual problem.** To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method. From the final simplex tableau, we then extract the solution to the original minimization problem.

Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its dual.

◆***Example 1*** Convert the following minimization problem into its dual.

Minimize $Z = 12x_1 + 16x_2$

Subject to: $x_1 + 2x_2 \geq 40$

$x_1 + x_2 \geq 30$

$x_1 \geq 0; \ x_2 \geq 0$

Solution: To achieve our goal, we first express our problem as the following matrix.

1	2	40
1	1	30
12	16	0

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a **transpose** of the original matrix. We get:

1	1	12
2	1	16
40	30	0

The following maximization problem associated with the above matrix is called its dual.

Maximize $Z = 40y_1 + 30y_2$

Subject to: $y_1 + y_2 \leq 12$

$2y_1 + y_2 \leq 16$

$y_1 \geq 0; \ y_2 \geq 0$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

◆***Example 2*** Solve graphically both the minimization problem and its dual maximization problem.

Solution: Our minimization problem is as follows.

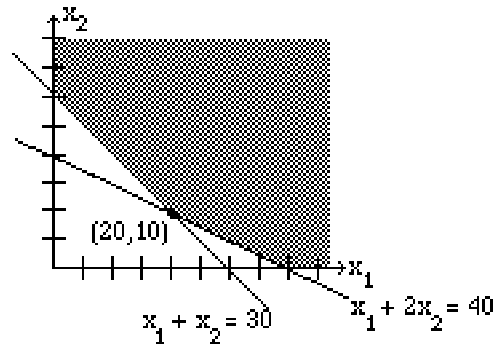
Minimize $Z = 12x_1 + 16x_2$

Subject to: $x_1 + 2x_2 \geq 40$

$x_1 + x_2 \geq 30$

$x_1 \geq 0;\ x_2 \geq 0$

We now graph the inequalities:



We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

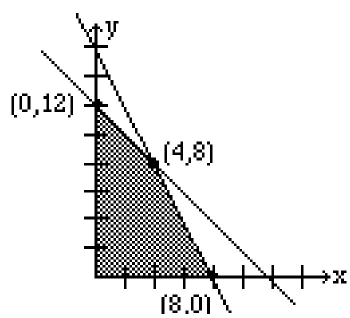
Now its dual is: **Maximize** $Z = 40y_1 + 30y_2$

Subject to: $y_1 + y_2 \leq 12$

$2y_1 + y_2 \leq 16$

$y_1 \geq 0;\ y_2 \geq 0$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The reader may recognize that Example 2 above is the same as Example 1, in section 3.1. It is also the same problem as Example 1 in section 4.1, where we solved it by the simplex method.

We observe that the minimum value of the minimization problem is the same as the maximum value of the maximization problem; in Example 2 the minimum and maximum are both 400. This is not a coincident. We state the duality principle.

The Duality Principle:
The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

Our next goal is to extract the solution for our minimization problem from the corresponding dual. To do this, we solve the dual by the simplex method.

◆***Example 3*** Find the solution to the minimization problem in Example 1 by solving its dual using the simplex method. We rewrite our problem.

Minimize $Z = 12x_1 + 16x_2$

Subject to: $x_1 + 2x_2 \geq 40$

$x_1 + x_2 \geq 30$

$x_1 \geq 0;\ x_2 \geq 0$

Solution: The dual is: **Maximize** $Z = 40y_1 + 30y_2$

Subject to: $y_1 + y_2 \leq 12$

$2y_1 + y_2 \leq 16$

$y_1 \geq 0;\ y_2 \geq 0$

Recall that we solved the above problem by the simplex method in Example 1, section 4.1. Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is

y_1	y_2	x_1	x_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are x_1 and x_2.

The final simplex tableau reads as follows:

y_1	y_2	x_1	x_2	Z	
0	1	2	–1	0	8
1	0	–1	1	0	4
0	0	20	10	1	400

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

y_1	y_2	x_1	x_2	Z	
0	1	2	–1	0	8
1	0	–1	1	0	4
0	0	20	10	1	400
		↑	↑		↑

We restate the solution as follows:

The minimization problem has a minimum value of 400 at the corner point (20, 10).

We now summarize our discussion.

MINIMIZATION BY THE SIMPLEX METHOD

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function as its bottom row.
3. Write the transpose of this matrix by interchanging the rows and columns.
4. Now write the dual problem associated with the transpose.
5. Solve the dual problem by the simplex method learned in section 4.1.
6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.

SECTION 4.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 1-2, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

1) Minimize $z = 6x_1 + 8x_2$

subject to $2x_1 + 3x_2 \geq 7$

$4x_1 + 5x_2 \geq 9$

$x_1, x_2 \geq 0$

---

2) Minimize $z = 5x_1 + 6x_2 + 7x_3$

subject to $3x_1 + 2x_2 + 3x_3 \geq 10$

$4x_1 + 3x_2 + 5x_3 \geq 12$

$x_1, x_2, x_3 \geq 0$

SECTION 4.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 3-4, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

3) Minimize $z = 4x_1 + 3x_2$

subject to

$$x_1 + x_2 \geq 10$$

$$3x_1 + 2x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

---

4) A diet is to contain at least 8 units of vitamins, 9 units of minerals, and 10 calories. Three foods, Food A, Food B, and Food C are to be purchased. Each unit of Food A provides 1 unit of vitamins, 1 unit of minerals, and 2 calories. Each unit of Food B provides 2 units of vitamins, 1 unit of minerals, and 1 calorie. Each unit of Food C provides 2 units of vitamins, 1 unit of minerals, and 2 calories. If Food A costs $3 per unit, Food B costs $2 per unit and Food C costs $3 per unit, how many units of each food should be purchased to keep costs at a minimum?

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

Solve the following linear programming problems using the simplex method.

1) Maximize $z = 5x_1 + 3x_2$ subject to $x_1 + x_2 \le 12$ $2x_1 + x_2 \le 16$ $x_1 \ge 0;\ x_2 \ge 0$	2) Maximize $z = 5x_1 + 8x_2$ subject to $x_1 + 2x_2 \le 30$ $3x_1 + x_2 \le 30$ $x_1 \ge 0;\ x_2 \ge 0$
3) Maximize $z = 2x_1 + 3x_2 + x_3$ subject to $4x_1 + 2x_2 + 5x_3 \le 32$ $2x_1 + 4x_2 + 3x_3 \le 28$ $x_1, x_2, x_3 \ge 0$	4) Maximize $z = x_1 + 6x_2 + 8x_3$ subject to $x_1 + 2x_2 \le 1200$ $2x_2 + x_3 \le 1800$ $4x_1 + x_3 \le 3600$ $x_1, x_2, x_3 \ge 0$
5) Maximize $z = 6x_1 + 8x_2 + 5x_3$ subject to $4x_1 + x_2 + x_3 \le 1800$ $2x_1 + 2x_2 + x_3 \le 2000$ $4x_1 + 2x_2 + x_3 \le 3200$ $x_1, x_2, x_3 \ge 0$	6) Minimize $z = 12x_1 + 10x_2$ subject to $x_1 + x_2 \ge 6$ $2x_1 + x_2 \ge 8$ $x_1 \ge 0;\ x_2 \ge 0$
7) Minimize $z = 4x_1 + 6x_2 + 7x_3$ subject to $x_1 + x_2 + 2x_3 \ge 20$ $x_1 + 2x_2 + x_3 \ge 30$ $x_1, x_2, x_3 \ge 0$	8) Minimize $z = 40x_1 + 48x_2 + 30x_3$ subject to $2x_1 + 2x_2 + x_3 \ge 25$ $x_1 + 3x_2 + 2x_3 \ge 30$ $x_1, x_2, x_3 \ge 0$

9) An appliance store sells three different types of ovens: small, medium, and large.
The small, medium, and large ovens require, respectively, 3, 5, and 6 cubic feet of storage space; a maximum of 1,000 cubic feet of storage space is available. Each oven takes 1hour of sales time; there is a maximum of 200 hours of sales labor time available for ovens. The small, medium, and large ovens require, respectively, 1, 1, and 2 hours of installation time; a maximum of 280 hours of installer labor for ovens is available monthly.
If the profit made from sales of small, medium and large ovens is $50, $100, and $150, respectively, how many of each type of oven should be sold to maximize profit, and what is the maximum profit?

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

10) A factory manufactures three products, A, B, and C. Each product requires the use of two machines, Machine I and Machine II. The total hours available, respectively, on Machine I and Machine II per month are 180 and 300. The time requirements and profit per unit for each product are listed below.

	A	B	C
Machine I	1	2	2
Machine II	2	2	4
Profit	20	30	40

How many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three products, A, B, and C, at its two factories, Factory I and Factory II. Daily production of each factory for each product is listed below.

	Factory I	Factory II
Product A	10	20
Product B	20	20
Product C	20	10

The company must produce at least 1000 units of product A, 1600 units of B, and 700 units of C. If the cost of operating Factory I is $4,000 per day and the cost of operating Factory II is $5000, how many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) For his classes, Professor Wright gives three types of quizzes, objective, recall, and recall-plus. To keep his students on their toes, he has decided to give at least 20 quizzes next quarter.

The three types, objective, recall, and recall-plus quizzes, require the students to spend, respectively, 10 minutes, 30 minutes, and 60 minutes for preparation, and Professor Wright would like them to spend at least 12 hours(720 minutes) preparing for these quizzes above and beyond the normal study time.

An average score on an objective quiz is 5, on a recall type 6, and on a recall-plus 7, and Dr. Wright would like the students to score at least 130 points on all quizzes.

It takes the professor one minute to grade an objective quiz, 2 minutes to grade a recall type quiz, and 3 minutes to grade a recall-plus quiz.

How many of each type should he give in order to minimize his grading time?

CHAPTER 5

Chapter 5: Exponential and Logarithmic Functions

In this chapter, you will

1. *examine exponential and logarithmic functions and their properties*
2. *identify exponential growth and decay functions and use them to model applications*
3. *use the natural base e to represent an exponential functions*
4. *use logarithmic functions to solve equations involving exponential functions*

In this chapter we examine exponential and logarithmic functions. We will need these functions in the next chapter, when examining financial calculations.

This chapter is a new addition to this textbook. The California Community Colleges Curriculum Course Descriptor for Finite Mathematics (C-ID; https://c-id.net/descriptors.html, http://www.ccccurriculum.net/articulation/) now requires coverage of exponential and logarithmic functions in a Finite Mathematics course that is part of an Associate Degree for Transfer.

Students enrolling in Finite Mathematics typically are required to complete an Intermediate Algebra course or equivalent, as a prerequisite, so students have already been exposed to much of the material in this chapter. However many students require a review of this material, which is the basis for financial calculations based on compound interest in the following chapter. In addition, review of this material is particularly important at colleges where Finite Mathematics serves as a prerequisite for Business Calculus.

This book assumes students have mastered working with exponents, and properties of exponents; it focuses on review of exponential and logarithmic functions with an eye toward skills needed to use exponential growth and decay models for financial calculations and other business applications, as well as subsequent use in a course on Business Calculus. For the most part, financial applications are not stressed in this new chapter, as financial calculations are the focus of the following chapter.

Students requiring more extensive review than provided here should refer to their algebra textbook. In addition, textbooks for Algebra, College Algebra, and Precalculus from OpenStax are available to use free online and to download for free at https://openstax.org/subjects/math

The material in this chapter derives from the following sources:

David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, "Chapter 4: Exponential and Logarithmic Functions," licensed under a Creative Commons CC BY-SA 3.0 license.

Precalculus textbook from OpenStax, which can be downloaded for free at https://openstax.org/details/precalculus , published under a Creative Commons Attribution License v4.0

Modifications of the above material are permitted under their respective CCA licenses.

In addition, this chapter includes material written by Roberta Bloom, Mathematics Instructor at De Anza College.

5.1 Exponential Growth and Decay Models

In this section, you will learn to

1. *recognize and model exponential growth and decay*
2. *compare linear and exponential growth*
3. *distinguish between exponential and power functions*

COMPARING EXPONENTIAL AND LINEAR GROWTH

Consider two social media sites which are expanding the number of users they have:

- Site A has 10,000 users, and expands by adding 1,500 new users each month
- Site B has 10,000 users, and expands by increasing the number of users by 10% each month.

The number of users for Site A can be modeled as linear growth. The number of users increases by a constant number, 1500, each month. If x = the number of months that have passed and y is the number of users, the number of users after x months is y = 10000+1500x.

For site B, the user base expands by a constant percent each month, rather than by a constant number. Growth that occurs at a constant percent each unit of time is called exponential growth.

We can look at growth for each site to understand the difference. The table compares the number of users for each site for 12 months. The table shows the calculations for the first 4 months only, but uses the same calculation process to complete the rest of the 12 months.

Month	Users at Site A	Users at Site B
0	10000	10000
1	10000 + 1500 = 11500	10000 + 10% of 10000 = 10000 + 0.10(10000) =10000(1.10) = 11000
2	11500 + 1500 = 13000	11000 + 10% of 11000 = 11000 + 0.10(11000) =11000(1.10) = 12100
3	13000 + 1500 = 14500	12100 + 10% of 12100 = 12100 + 0.10(12100) =12100(1.10) = 13310
4	14500 + 1500 = 16000	13310 + 10% of 13310 = 13310 + 0.10(13310) =13310(1.10) = 14641
5	17500	16105
6	19000	17716
7	20500	19487
8	22000	21436
9	23500	23579
10	25000	25937
11	26500	28531
12	28000	31384

For Site B, we can re-express the calculations to help us observe the patterns and develop a formula for the number of users after x months.

Month 1: $y = 10000(1.1) = 11000$

Month 2: $y = 11000(1.1) = 10000(1.1)(1.1) = \mathbf{10000(1.1)^2} = 12100$

Month 3: $y = 12100(1.1) = 10000(1.1)^2(1.1) = \mathbf{10000(1.1)^3} = 13310$

Month 4: $y = 13310(1.1) = 10000(1.1)^3(1.1) = \mathbf{10000(1.1)^4} = 14641$

By looking at the patterns in the calculations for months 2, 3, and 4, we can generalize the formula. After x months, the number of users y is given by the function $\mathbf{y = 10000(1.1)^x}$

USING EXPONENTIAL FUNCTIONS TO MODEL GROWTH AND DECAY

In exponential growth, the value of the dependent variable y increases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential growth functions include:

- the number of residents of a city or nation that grows at a constant percent rate.
- the amount of money in a bank account that earns interest if money is deposited at a single point in time and left in the bank to compound without any withdrawals.

In exponential decay, the value of the dependent variable y decreases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential decay functions include:

- value of a car or equipment that depreciates at a constant percent rate over time
- the amount a drug that still remains in the body as time passes after it is ingested
- the amount of radioactive material remaining over time as a radioactive substance decays.

Exponential functions often model quantities as a function of time; thus we often use the letter t as the independent variable instead of x.

The table compares exponential growth and exponential decay functions:

Exponential Growth	**Exponential Decay**
Quantity grows by a constant percent per unit of time	Quantity decreases by a constant percent per unit of time
$\mathbf{y = ab^x}$ • a is a positive number representing the initial value of the function when $x = 0$ • b is a real number that is greater than 1: $b > 1$ • the growth rate r is a positive number, $r > 0$ where $b = 1 + r$ (so that $r = b - 1$)	$\mathbf{y = ab^x}$ • a is a positive number representing the initial value of the function when $x = 0$ • b is a real number that is between 0 and 1: $0 < b < 1$ • the decay rate r is a negative number, $r < 0$ where $b = 1 + r$ (so that $r = b - 1$)

In general, the domain of exponential functions is the set of all real numbers.
The range of an exponential growth or decay function is the set of all positive real numbers.

In most applications, the independent variable, x or t, represents time. When the independent variable represents time, we may choose to restrict the domain so that independent variable can have only non-negative values in order for the application to make sense. If we restrict the domain, then the range is also restricted as well.

- For an exponential growth function $y = ab^x$ with $b>1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $y \geq a$.
- For an exponential decay function $y = ab^x$ with $0<b<1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $0 < y \leq a$.

◆***Example 1*** Consider the growth models for social media sites A and B, where x = number of months since the site was started and y = number of users.
The number of users for Site A follows the linear growth model:

$$y = 10000+1500x.$$

The number of users for Site B follows the exponential growth model:

$$y = 10000(1.1^x)$$

For each site, use the function to calculate the number of users at the end of the first year, to verify the values in the table. Then use the functions to predict the number of users after 30 months.

Solution: Since x is measured in months, then x = 12 at the end of one year.

Linear Growth Model:

When x = 12 months, then y = 10000 + 1500(12) = 28,000 users
When x = 30 months, then y = 10000 + 1500(30) = 55,000 users

Exponential Growth Model:

When x = 12 months, then $y = 10000(1.1^{12}) = 31{,}384$ users
When x = 30 months, then $y = 10000(1.1^{30}) =174{,}494$ users

We see that as x, the number of months, gets larger, the exponential growth function grows large faster than the linear function (even though in Example 1 the linear function initially grew faster). This is an important characteristic of exponential growth: exponential growth functions always grow faster and larger in the long run than linear growth functions.

It is helpful to use function notation, writing $y = f(t) = ab^t$, to specify the value of t at which the function is evaluated.

◆***Example 2*** A forest has a population of 2000 squirrels that is increasing at the rate of 3% per year. Let t = number of years and y = f(t) = number of squirrels at time t.
a. Find the exponential growth function that models the number of squirrels in the forest at the end of t years.
b. Use the function to find the number of squirrels after 5 years and after 10 years

Solution: a. The exponential growth function is $y = f(t) = ab^t$, where $a = 2000$ because the initial population is 2000 squirrels

The annual growth rate is 3% per year, stated in the problem. We will express this in decimal form as $r = 0.03$

Then $b = 1+r = 1+0.03 = 1.03$

Answer: The exponential growth function is $y = f(t) = 2000(1.03^t)$

b. After 5 years, the squirrel population is $y = f(5) = 2000(1.03^5) \approx 2319$ squirrels

After 10 years, the squirrel population is $y = f(10) = 2000(1.03^{10}) \approx 2688$ squirrels

◆***Example 3*** A large lake has a population of 1000 frogs. Unfortunately the frog population is decreasing at the rate of 5% per year.
Let t = number of years and $y = g(t)$ = the number of frogs in the lake at time t.
a. Find the exponential decay function that models the population of frogs.
b. Calculate the size of the frog population after 10 years.

Solution: a. The exponential decay function is $y = g(t) = ab^t$, where $a = 1000$ because the initial population is 1000 frogs

The annual decay rate is 5% per year, stated in the problem. The words decrease and decay indicated that r is negative. We express this as $r = -0.05$ in decimal form.

Then, $b = 1 + r = 1 + (-0.05) = 0.95$

Answer: The exponential decay function is: $y = g(t) = 1000(0.95^t)$

b. After 10 years, the frog population is $y = g(10) = 1000(0.95^{10}) \approx 599$ frogs

◆***Example 4*** A population of bacteria is given by the function $y = f(t) = 100(2^t)$, where t is time measured in hours and y is the number of bacteria in the population.
a. What is the initial population?
b. What happens to the population in the first hour?
c. How long does it take for the population to reach 800 bacteria?

Solution: a. The initial population is 100 bacteria. We know this because $a = 100$ and because at time $t = 0$, then $f(0) = 100(2^0) = 100(1)=100$

b. At the end of 1 hour, the population is $y = f(1) = 100(2^1) = 100(2)=200$ bacteria. The population has doubled during the first hour.

c. We need to find the time t at which $f(t) = 800$. Substitute 800 as the value of y:

$$y = f(t) =100(2^t)$$

$$800 =100(2^t)$$

Divide both sides by 100 to isolate the exponential expression on the one side

$$8 = 1(2^t)$$

$8 = 2^3$, so it takes $t = 3$ hours for the population to reach 800 bacteria.

Two important notes about Example 4:

- In solving $8 = 2^t$, we "knew" that t is 3. But we usually can **not** know the value of the variable just by looking at the equation. Later we will use logarithms to solve equations that have the variable in the exponent.
- To solve $800 = 100(2^t)$, we divided both sides by 100 to isolate the exponential expression 2^t. We can not multiply 100 by 2. Even if we write it as $800 = 100(2)^t$, which is equivalent, we still can **not** multiply 100 by 2. The exponent applies **only** to the quantity immediately before it, so the exponent t applies only to the base of 2.

COMPARING LINEAR, EXPONENTIAL AND POWER FUNCTIONS

To identify the type of function from its formula, we need to carefully note the position that the variable occupies in the formula.

A linear function can be written in the form y = ax + b.
As we studied in chapter 1, there are other forms in which linear equations can be written, but linear functions can all be rearranged to have form y = mx + b.

An exponential function has form $y = ab^x$
The variable x is in the exponent. The base b is a positive number.
If b>1, the function represents exponential growth.
If 0 < b < 1, the function represents exponential decay

A power function has form $y = cx^p$
The variable x is in the base. The exponent p is a non-zero number.

We compare three functions:

- linear function y = f(x) = 2x
- exponential function $y = g(x) = 2^x$
- power function $y = h(x) = x^2$

x	**y = f(x) =2x**	**$y = g(x) = 2^x$**	**$y = h(x) = x^2$**
0	0	1	0
1	2	2	1
2	4	4	4
3	6	8	9
4	8	16	16
5	10	32	25
6	12	64	36
10	20	1024	100
Type of function	Linear y = mx+b	Exponential $y = ab^x$	Power $y = cx^p$
How to recognize equation for this type of function.	all terms are first degree; m is slope; b is the y intercept	base is a number b>0; the variable is in the exponent	variable is in the base; exponent is a number $p \neq 0$
	For equal intervals of change in x, y increases by a constant amount	For equal intervals of change in x, y increases by a constant ratio	

For the functions in the previous table: linear function $y = f(x) = 2x$, exponential function $y = g(x) = 2^x$, and power function $y = h(x) = x^2$, if we restrict the domain to $x \geq 0$ only, then all these functions are growth functions. When $x \geq 0$, the value of y increases as the value of x increases.

The exponential growth function grows large faster than the linear and power functions, as x gets large. This is always true of exponential growth functions, as x gets large enough.

◆**Example 5** Classify the functions below as exponential, linear, or power functions.

a. $y=10x^3$ b. $y=1000-30x$ c. $y=1000(1.05^x)$

d. $y=500(0.75^x)$ e. $y=10\sqrt[3]{x} = x^{1/3}$ f. $y=5x-1$ g. $y=6/x^2 = 6x^{-2}$

Solution: The exponential functions are

c. $y=1000(1.05^x)$ The variable is in the exponent; the base is the number b = 1.05

d. $y=500(0.75^x)$ The variable is in the exponent; the base is the number b = 0.75

The linear functions are b. $y=1000-30x$ and f. $y=5x-1$

The power functions are

a. $y=10x^3$ The variable is the base; the exponent is a fixed number, p=3.

e. $y=10\sqrt[3]{x} = 10x^{1/3}$ The variable is the base; the exponent is a number, p=1/3.

g. $y=6/x^2 = 6x^{-2}$ The variable is the base; the exponent is a number, p = –2.

NATURAL BASE: e

The number *e* is often used as the base of an exponential function. *e* is called the natural base.

***e* is approximately 2.71828**

e is an irrational number with an infinite number of decimals; the decimal pattern never repeats.

Section 6.2 includes an example that shows how the value of *e* is developed and why this number is mathematically important. Students studying Finite Math should already be familiar with the number *e* from their prerequisite algebra classes.

When *e* is the base in an exponential growth or decay function, it is referred to as **continuous growth or continuous decay**. We will use *e* in chapter 6 in financial calculations when we examine interest that compounds continuously.

Any exponential function can be written in the form $y = ae^{kx}$

k is called the continuous growth or decay rate.

- If $k > 0$, the function represents exponential growth
- If $k < 0$, the function represents exponential decay

a is the initial value

We can rewrite the function in the form $y = ab^x$, where $b=e^k$

In general, if we know one form of the equation, we can find the other forms. For now, we have not yet covered the skills to find k when we know b. After we learn about logarithms later in this chapter, we will find k using natural log: $k = \ln b$.

The table below summarizes the forms of exponential growth and decay functions.

	$y = ab^x$	**$y = a(1+r)^x$**	**$y = ae^{kx}$, $k \neq 0$**
Initial value	$a>0$	$a>0$	$a>0$
Relationship between b, r, k	$b > 0$	$b=1+ r$	$b = e^k$ and $k = \ln b$
Growth	$b > 1$	$r > 0$	$k > 0$
Decay	$0 < b < 1$	$r < 0$	$k < 0$

◆***Example 6*** The value of houses in a city are increasing at a continuous growth rate of 6% per year. For a house that currently costs \$400,000:

a. Write the exponential growth function in the form $y=ae^{kx}$.

b. What would be the value of this house 4 years from now?

c. Rewrite the exponential growth function in the form $y=ab^x$.

d. Find and interpret r.

Solution: a. The initial value of the house is a = \$400000

The problem states that the **continuous** growth rate is 6% per year, so $k = 0.06$

The growth function is : $y=400000e^{0.06x}$

b. After 4 years, the value of the house is $y=400000e^{0.06\,(4)}$ = \$508,500.

c. To rewrite $y=400000e^{0.06x}$ in the form $y = ab^x$, we use the fact that $b=e^k$.

$b=e^{0.06}$

$b=1.06183657\approx1.0618$

$y = 400000(1.0618)^x$

d. To find r, we use the fact that $b=1+ r$

$b=1.0618$

$1+ r =1.0618$

$r =0.0618$

The value of the house is **increasing** at an **annual rate** of 6.18%.

◆***Example 7*** Suppose that the value of a certain model of new car decreases at a continuous decay rate of 8% per year. For a car that costs $20,000 when new:

a. Write the exponential decay function in the form $y=ae^{kx}$.

b. What would be the value of this car 5 years from now?

c. Rewrite the exponential decay function in the form $y=ab^x$.

d. Find and interpret r.

Solution: a. The initial value of the car is a = $20000

The problem states that the **continuous** decay rate is 8% per year, so k = –0.08

The growth function is : $y=20000e^{-0.08x}$

b. After 5 years, the value of the car is $y=20000\,e^{-0.08\,(5)}$ = $13,406.40.

c. To rewrite $y=20000e^{-0.08x}$ in the form $y = ab^x$, we use the fact that $b=e^k$.

$b=e^{-0.08}$

$b=0.9231163464\approx0.9231$

$y = 20000(0.9231)^x$

d. To find r, we use the fact that b=1+ r

b=0.9231

1+ r =0.9231

r =0.9231–1= –0.0769

The value of the car is **decreasing** at an **annual rate** of 7.69%.

SECTION 5.1 PROBLEM SET: EXPONENTIAL GROWTH AND DECAY FUNCTIONS

Identify each as an exponential, linear, or power function

1) $y = 640\ (1.25^x)$	2) $y = 640\ (x^{1.25})$
3) $y = 640\ (1.25x)$	4) $y = 1.05x - 2.5$
5) $y = 90 - (4/5)x$	6) $y = 42(0.92^x)$
7) $y = 37(x^{0.25})$	8) $y = 4(1/3)^x$

Indicate if the function represents exponential growth or exponential decay.

9) $y = 127e^{-0.35t}$	10) $y = 70\ (0.8^t)$
11) $y = 453(1.2^t)$	12) $y = 16e^{0.2t}$

In each of the following, y is an exponential function of t stated in the form $y = ae^{kt}$ where t represents time measured in years.. For each:

a. re-express each function in the form $y = ab^t$ (state the value of b accurate to 4 deicmal places)
b. state the annual growth rate or annual decay rate as a percent, accurate to 2 decimal places

13) $y = 127e^{-0.35t}$	14) $y = 16e^{0.4t}$
15) $y = 17250\ e^{0.24t}$	16) $y = 4700\ e^{-0.07t}$

SECTION 5.1 PROBLEM SET: EXPONENTIAL GROWTH AND DECAY FUNCTIONS

Identify if the function represents exponential growth, exponential decay, linear growth, or linear decay. In each case write the function and find the value at the indicated time.

17) A house was purchased for \$350,000 in the year 2010. The value has been increasing by \$7,000 per year. Write the function and find the value of the house after 5 years.	18) A house was purchased for \$350,000 in the year 2010. The value has been increasing at the rate of 2% per year. Write the function and find the value of the house after 5 years.
19) A lab purchases new equipment for \$50,000. Its value depreciates over time. The value decreases at the rate of 6% annually. Write the function and find the value after 10 years.	20) A lab purchases new equipment for \$50,000. Its value depreciates over time. The value decreases by \$3000 annually. Write the function and find the value after 10 years
21) A population of bats in a cave has 200 bats. The population is increasing by 10 bats annually. Write the function. How many bats live in the cave after 7 years?	22) A population of bats in a cave has 200 bats. The population is increasing at the rate of 5% annually. Write the function. How many bats live in the cave after 7 years?
23) A population of a certain species of bird in a state park has 300 birds. The population is decreasing at the rate of 7% year. Write the function. How many birds are in the population after 6 years?	24) A population of a certain species of bird in a state park has 300 birds. The population decreases by 20 birds per year. Write the function. How many birds are in the population after 6 years?

SECTION 5.1 PROBLEM SET: EXPONENTIAL GROWTH AND DECAY FUNCTIONS

In problems 25-28, the problem represents exponential growth or decay and states the CONTINUOUS growth rate or continuous decay rate. Write the exponential growth or decay function and find the value at the indicated time.

Hint: Use the form of the exponential function that is appropriate when the CONTINUOUS growth or decay rate is given.

25) A population of 400 microbes increases at the **continuous** growth rate of 26% per day. Write the function and find the number of microbes in the population at the end of 7 days.	26) The price of a machine needed by a production factory is $28,000. The business expects to replace the machine in 4 years. Due to inflation the price of the machine is increasing at the **continuous** rate of 3.5% per year. Write the function and find the value of the machine 4 years from now.
27) A population of an endangered species consists of 4000 animals of that species. The population is decreasing at the continuous rate of 12% per year. Write the function and find the size of the population at the end of 10 years.	28) A business buys a computer system for $12000. The value of the system is depreciating and decreases at the continuous rate of 20% per year. Write the function and find the value at the end of 3 years.

5.2 Graphs and Properties of Exponential Growth and Decay Functions

In this section, you will:

1. *examine properties of exponential functions*
2. *examine graphs of exponential functions*

An exponential function can be written in forms $f(x) = ab^x = a(1 + r)^x = ae^{kx}$

a is the initial value because $f(0) = a$.
In the growth and decay models that we examine in this finite math textbook, $a > 0$.

b is often called the growth factor. We restrict b to be positive ($b > 0$) because even roots of negative numbers are undefined. We want the function to be defined for all values of x, but b^x would be undefined for some values of x if $b<0$.

r is called the growth or decay rate. In the formula for the functions, we use r in decimal form, but in the context of a problem we usually state r as a percent.

k is called the continuous growth rate or continuous decay rate.

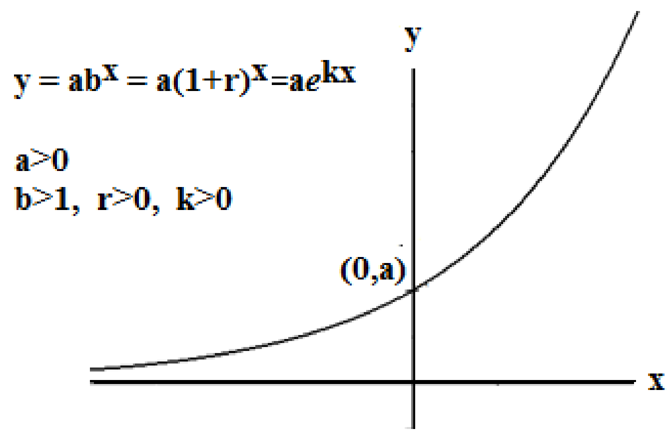
PROPERTIES OF EXPONENTIAL GROWTH FUNCTIONS

The function $y=f(x) = ab^x$ represents growth if $b > 1$ and $a > 0$.

The growth rate r is positive when $b>1$. Because $b = 1+ r > 1$, then $r = b–1 > 0$

The function $y=f(x) = ae^{kx}$ represents growth if $k > 0$ and $a > 0$.

The function is an increasing function; y increases as x increases.



Domain: { all real numbers} ; all real numbers can be input to an exponential function

Range: If $a>0$, the range is {positive real numbers}
The graph is always above the x axis.

Horizontal Asymptote: when $b > 1$, the horizontal asymptote is the negative x axis, as x becomes large negative. Using mathematical notation: as $x \to -\infty$, then $y \to 0$.

The vertical intercept is the point (0,a) on the y-axis.
There is no horizontal intercept because the function does not cross the x-axis.

PROPERTIES OF EXPONENTIAL DECAY FUNCTIONS

The function $y=f(x) = ab^x$ function represents decay if $0 < b < 1$ and $a > 0$.

The growth rate r is negative when $0 < b<1$. Because $b = 1+ r < 1$, then $r = b–1 < 0$.

The function $y=f(x) = ae^{kx}$ function represents decay if $k < 0$ and $a > 0$.

The function is a decreasing function; y decreases as x increases.

Domain: { all real numbers} ; all real numbers can be input to an exponential function

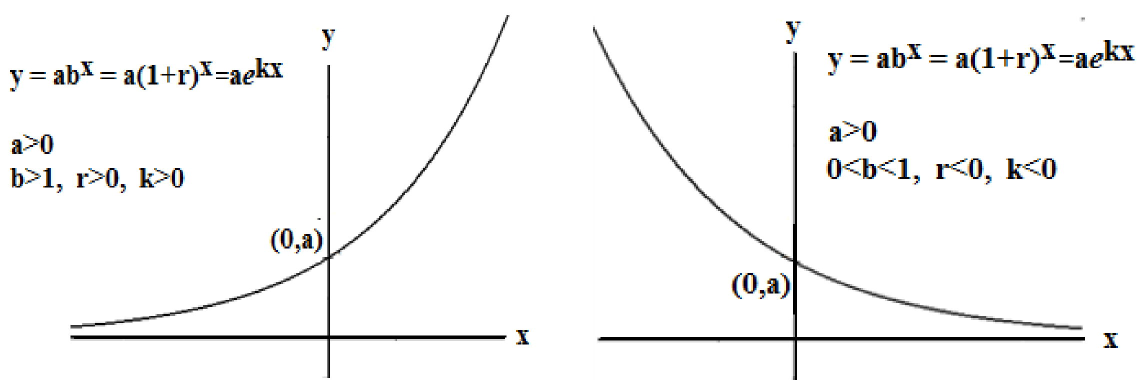
Range: If a>0, the range is {positive real numbers}
The graph is always above the x axis.

Horizontal Asymptote: when $b < 1$, the horizontal asymptote is the positive x axis as x becomes large positive. Using mathematical notation: as $x \to \infty$, then $y \to 0$

The vertical intercept is the point (0,a) on the y-axis.
There is no horizontal intercept because the function does not cross the x-axis.

The graphs for exponential growth and decay functions are displayed below for comparison

EXPONENTIAL GROWTH EXPONENTIAL DECAY



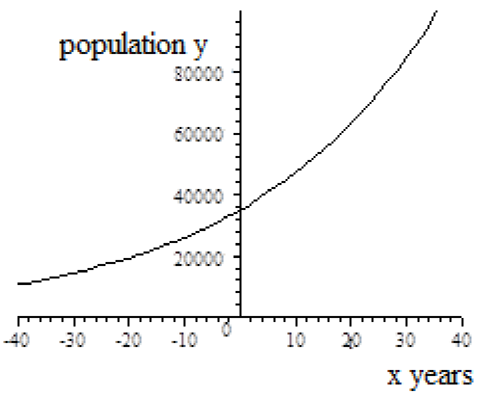
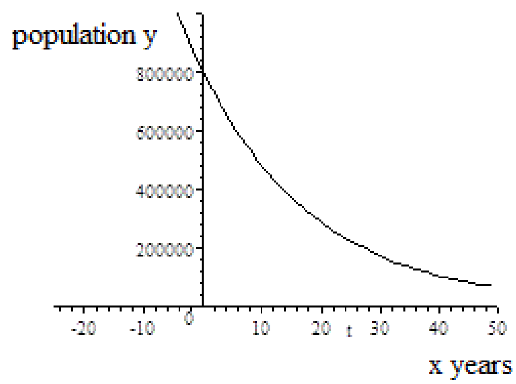
AN EXPONENTIAL FUNCTION IS A ONE-TO-ONE FUNCTION

Observe that in the graph of an exponential function, each y value on the graph occurs only once. Therefore, every y value in the range corresponds to only one x value. So, for any particular value of y, you can use the graph to see which value of x is the input to produce that y value as output. This property is called "**one-to-one**".

Because for each value of the output y, you can uniquely determine the value of the corresponding input x, thus every exponential function has an inverse function.
The inverse function of an exponential function is a logarithmic function, which we will investigate in the next section.

◆***Example 1*** x years after the year 2015, the population of the city of Fulton is given by the function $y= f(x) = 35000(1.03^x)$.
x years after the year 2015, the population of the city of Greenville is given by the function $y = g(x) = 80000(0.95^x)$
Compare the graphs of these functions.

Solution: The graphs below were created using computer graphing software.
You can also graph these functions using a graphing calculator.

Population of Fulton	Population of Greenville
$y= f(x) = 35000(1.03^x)$	$y= g(x) = 80000(0.95^x)$
Fulton's population is increasing. $b = 1.03 >1$ and $r = 0.03 > 0$ Exponential Growth	Greenville's population is decreasing. $b = 0.95 < 1$ and $r = –0.05 < 0$ Exponential Decay
	
y-intercept: (0, 35000) The initial population in 2015 is 35000.	**y-intercept:** (0,80000) The initial population in 2015 is 80000.
Horizontal Asymptote: The negative x axis is the horizontal asymptote. $y \to 0$ as $x \to -\infty$	**Horizontal Asymptote:** The positive x axis is the horizontal asymptote. $y \to 0$ as $x \to \infty$

Domain: In general, the domain of both functions $y= f(x) = 35000(1.03^x)$ and $y= g(x) = 80000(0.95^x)$ is the set of all real numbers.

Range: The range of both functions is the set of positive real numbe rs. Both graphs always lie above the x-axis.

Domain and Range in context of this problem:
The functions represent population size as a function of time ***after*** the year 2015 .
We restrict the domain in this context, using the "practical domain" as the set of all non-negative real numbers: $x\geq 0$. Then we would consider only the portion of the graph that lies in the first quadrant.

- If we restrict the domain to $x\geq 0$ for the growth function $y= f(x) = 35000(1.03^x)$, then the range for the population of Fulton is $y \geq 35{,}000$
- If we restrict the domain to $x\geq 0$ for the decay function $y= g(x) = 80000(0.95^x)$, then the range for the population of Greenville is $y \leq 80{,}000$.

SECTION 5.2 PROBLEM SET: GRAPHS AND PROPERTIES OF EXPONENTIAL GROWTH AND DECAY FUNCTIONS

In questions 1-4, let t = time in years and y = the value at time t or y = the size of the population at time t. The domain is the set of non-negative values for t; $t \geq 0$, because y represents a physical quantity and negative values for time may not make sense. For each question:

a. Write the formula for the function in the form $y = ab^t$

b. Sketch the graph of the function and mark the coordinates of the y-intercept.

1) A house was purchased for $350,000 in the year 2010. The value has been increasing at the rate of 2% per year.	2) A population of a certain species of bird in a state park has 300 birds. The population is decreasing at the rate of 7% year.
3) A lab buys equipment $50,000. Its value depreciates over time. The value decreases at the rate of 6% annually.	4) A population of bats in a cave has 200 bats. The population is increasing at the rate of 5% annually.

SECTION 5.2 PROBLEM SET: GRAPHS AND PROPERTIES OF EXPONENTIAL GROWTH AND DECAY FUNCTIONS

In questions 5-8, let t = time in years and y = the value at time t or y = the size of the population at time t. The domain is the set of non-negative values for t; $t \geq 0$, because y represents a physical quantity and negative values for time may not make sense. For each question:

a. Write the formula for the function in the form $y = ae^{kt}$

b. Sketch the graph of the function and mark the coordinates of the y-intercept.

5) A population of 400 microbes increases at the **continuous** growth rate of 26% per day.	6) The price of a machine needed by a production factory is $28,000. Due to inflation the price of the machine is increasing at the **continuous** rate of 3.5% per year.
7) A population of an endangered species consists of 4000 animals of that species. The population is decreasing at the **continuous** rate of 12% per year.	8) A business buys a computer system for $12000. The value of the system is depreciating and decreases at the **continuous** rate of 20% per year.

SECTION 5.2 PROBLEM SET: GRAPHS AND PROPERTIES OF EXPONENTIAL GROWTH AND DECAY FUNCTIONS

For questions 9-12

a. sketch a graph of exponential function

b. list the coordinates of the y intercept

c. state the equation of any asymptotes and state the whether the function approaches the asymptote as $x \to \infty$ or as $x \to -\infty$

d. State the domain and range

9) $y = 10(1.5^x)$	6) $y = 10(e^{1.2x})$
11) $y = 32(0.75^x)$	12) $y = 200(e^{-.5x})$

5.3 Logarithms and Logarithmic Functions

In this section you will learn

1. the definition of logarithmic function as the inverse of the exponential function
2. to write equivalent logarithmic and exponential expressions
3. the definition of common log and natural log
4. properties of logs
5. to evaluate logs using the change of base formula

THE LOGARITHM

Suppose that a population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where x represents the number of weeks that have passed. When will this population reach 500?

Trying to solve this problem leads to $500 = 50(2)^x$

Dividing both sides by 50 to isolate the exponential leads to $10 = 2^x$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution, but we would like to find an algebraic way to solve the equation.

We need an inverse operation to exponentiation in order to solve for the variable if the variable is in the exponent. As we learned in algebra class (prerequisite to this finite math course), the inverse function for an exponential function is a logarithmic function.

We also learned that an exponential function has an inverse function, because each output (y) value corresponds to only one input (x) value. The name given this property was “one-to-one”.

If you need to review the concept of an inverse function and how to determine if a function is one-to-one and has an inverse, you can refer to your algebra textbook or to College Algebra from Openstax available free online at https://openstax.org/details/college-algebra

Logarithm

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

y = log$_b$(x) is equivalent to b^y = x

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: $b>0$

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$\log_b(b^x) = x$ and $b^{\log_b(x)} = x$

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

◆***Example 1*** Write these exponential equations as logarithmic equations:

a. $2^3 = 8$ b. $5^2 = 25$ c. $10^{-3} = \dfrac{1}{1000}$

Solution:

a. $2^3 = 8$ can be written as a logarithmic equation as $\log_2(8) = 3$

b. $5^2 = 25$ can be written as a logarithmic equation as $\log_5(25) = 2$

c. $10^{-3} = \dfrac{1}{1000}$ can be written as a logarithmic equation as $\log_{10}\left(\dfrac{1}{1000}\right) = -3$

◆***Example 2*** Write these logarithmic equations as exponential equations:

a. $\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ b. $\log_3(9) = 2$

Solution:

a. $\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ can be written as an exponential equation as $6^{\frac{1}{2}} = \sqrt{6}$

b. $\log_3(9) = 2$ can be written as an exponential equation as $3^2 = 9$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

◆***Example 3*** Solve $\log_4(x) = 2$ for x.

Solution: By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$

◆***Example 4*** Solve $2^x = 10$ for x.

Solution: By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful—often we really need a decimal approximation to the solution. Luckily, this is a task that calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases: base 10 and base *e*. Happily, this ends up not being a problem, as we'll see soon that we can use a "change of base" formula to evaluate logarithms for other bases.

COMMON AND NATURAL LOGARITHMS

The **common log** is the logarithm with base 10, and is typically written log (x).
If the base is not indicated in the log function, then the base b used is b=10.

The **natural log** is the logarithm with base e, and is typically written ln (x).

Note that for any other base b, other than 10, the base must be indicated in the notation $\log_b(x)$

◆***Example 5*** Evaluate log(1000) using the definition of the common log.

Solution: The table shows values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	–1
0.01	10^{-2}	–2
0.001	10^{-3}	–3

To evaluate log(1000), we can say

$x = \log(1000)$

Then rewrite the equation in exponential form using the common log base of 10

$10^x = 1000$

From this, we might recognize that 1000 is the cube of 10, so

$x = 3$.

Alternatively, we can use the inverse property of logs to write

$\log_{10}(10^3) = 3$

◆***Example 6*** Evaluate log(1/1,000,000)

Solution: To evaluate log(1/1,000,000), we can say

$x = \log(1/1{,}000{,}000) = \log(1/10^6) = \log(10^{-6})$

Then rewrite the equation in exponential form: $10^x = 10^{-6}$

Therefore x = –6

Alternatively, we can use the inverse property of logs to find the answer:

$\log_{10}(10^{-6}) = -6$

◆***Example 7*** Evaluate a. $\ln e^5$ b. $\ln\sqrt{e}$.

Solution: a. To evaluate $\ln e^5$, we can say

$x = \ln e^5$

Then rewrite into exponential form using the natural log base of e

$e^x = e^5$

Therefore x = 5

Alternatively, we can use the inverse property of logs to write $\ln(e^5) = 5$

b. To evaluate $\ln\sqrt{e}$, we recall that roots are represented by fractional exponents

$$x = \ln\sqrt{e} = \ln\left(\sqrt{e}\right) = \ln\left(e^{1/2}\right)$$

Then rewrite into exponential form using the natural log base of e

$e^x = e^{1/2}$

Therefore x = 1/2

Alternatively, we can use the inverse property of logs to write $\ln(e^{1/2}) = 1/2$

◆***Example 8*** Evaluate the following using your calculator or computer:

a. log 500 b. ln 500

Solution: a. Using the LOG key on the calculator to evaluate logarithms in base 10, we evaluate LOG(500)

Answer: $\log 500 \approx 2.69897$

b. Using the LN key on the calculator to evaluate natural logarithms, we evaluate LN(500)

Answer: $\ln 500 \approx 6.214608$

SOME PROPERTIES OF LOGARITHMS

We often need to evaluate logarithms using a base other than 10 or *e*. To find a way to utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need some additional properties.

Properties of Logs: Exponential Property : $\log_b(A^q) = q\log_b(A)$

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$$ for any bases b, c >0

To show why these properties are true, we offer proofs.

Proof of Exponent Property: $\log_b(A^q) = q\log_b(A)$

Since the logarithmic and exponential functions are inverses,

$\log_b(A^q) = \mathrm{A}$

So $A^q = \left(b^{\log_b A}\right)^q$

Utilizing the exponential rule that states $\left(x^p\right)^q = x^{pq}$, we get

$A^q = \left(b^{\log_b A}\right)^q = b^{q\log_b A}$

Then $\log_b A^q = \log_b b^{q\log_b A}$

Again utilizing the inverse property on the right side yields the result

$\log_b A^q = q\log_b A$

Proof of Change of Base Property: $\log_b(A) = \dfrac{\log_c(A)}{\log_c(b)}$ for any bases b, c >0

Let $\log_b(A) = x$.

Rewriting as an exponential gives $b^x = A$.

Taking the log base c of both sides of this equation gives $\log_c b^x = \log_c A$.

Now utilizing the exponent property for logs on the left side,

$x \log_c b = \log_c A$

Dividing, we obtain $x = \dfrac{\log_c(A)}{\log_c(b)}$ which is the change of base formula.

EVALUATING LOGARITHMS

With the change of base formula, $\log_b(A) = \dfrac{\log_c(A)}{\log_c(b)}$ for any bases b, c >0,

we can finally find a decimal approximation to our question from the beginning of the section.

◆***Example 9*** Solve $2^x = 10$ for x.

Solution: Rewrite exponential equation $2^x = 10$ as a logarithmic equation

$\mathrm{x} = \log_2(10)$

Using the change of base formula, we can rewrite log base 2 as a logarithm of any other base. Since our calculators can evaluate natural log, we can choose to use the natural logarithm, which is the log base e:

Using our calculators to evaluate this, $\dfrac{\ln(10)}{\ln(2)}$ = LN(10)/LN(2) ≈ 3.3219

This finally allows us to answer our original question from the beginning of this section: For the population of 50 flies that doubles every week, it will take approximately 3.32 weeks to grow to 500 flies.

◆***Example 10*** Evaluate $\log_5(100)$ using the change of base formula.

Solution: We can rewrite this expression using any other base.

Method 1: We can use natural logarithm base *e* with the change of base formula

$$\log_5(100) = \frac{\ln(100)}{\ln(5)} = \text{LN}(100)/\text{LN}(5) \approx 2.861$$

Method 2: We can use common logarithm base 10 with the change of base formula,

$$\log_5(100) = \frac{\log(100)}{\log(5)} = \text{LOG}(100)/\text{LOG}(5) \approx 2.861$$

We summarize the relationship between exponential and logarithmic functions

Logarithms

The logarithm (base *b*) function, written $\log_b(x)$, is the inverse of the exponential function (base *b*), b^x.

$y = \log_b(x)$ is equivalent to $b^y = x$

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: b>0

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$\log_b(b^x) = x$ and $b^{\log_b(x)} = x$

Properties of Logs: Exponential Property: $\log_b(A^q) = q\log_b(A)$

Properties of Logs: Change of Base

$\log_b(A) = \dfrac{\log_c(A)}{\log_c(b)}$ for any base b, c >0

The inverse, exponential and change of base properties above will allow us to solve the equations that arise in problems we encounter in this textbook. For completeness, we state a few more properties of logarithms

Sum of Logs Property: $\log_b(A) + \log_b(C) = \log_b(AC)$

Difference of Logs Property: $\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right)$

Logs of Reciprocals: $\log_b\left(\frac{1}{C}\right) = -\log_b(C)$

Reciprocal Bases: $\log_{1/b} C = -\log_b(C)$

SECTION 5.3 PROBLEM SET: LOGARITHMS AND LOGARITHMIC FUNCTIONS

Rewrite each of these exponential expressions in logarithmic form:

1) $3^4=81$	2) $10^5=100{,}000$
3) $5^{-2}=0.04$	4) $4^{-1}=0.25$
5) $16^{1/4}=2$	6) $9^{1/2}=3$

Rewrite each of these logarithmic expressions in exponential form:

7) $\log_5 625 = 4$	8) $\log_2 (1/32) = -5$
9) $\log_{11} 1331 = 3$	10) $\log_{10} 0.0001 = -4$
11) $\log_{64} 4 = 1/3$	12) $\ln\sqrt{e} = \frac{1}{2}$

If the expression is in exponential form, rewrite it in logarithmic form.
If the expression is in logarithmic form, rewrite it in exponential form.

13) $5^x=15625$	14) $x = 9^3$
15) $\log_5 125 = x$	16) $\log_3 x = 5$
17) $\log_{10} y = 4$	18) $e^x = 10$
19) $\ln x = -1$	20) $e^5 = y$

SECTION 5.3 PROBLEM SET: LOGARITHMS AND LOGARITHMIC FUNCTIONS

For each equation, rewrite in exponential form and solve for x.

21) $\log_5 (x) = 3$	22) $\log_2 (x) = -2$
23) $\log_{10} (x) = -3$	24) $\log_3 (x) = 6$
25) $\log_{25} (x) = 1/2$	26) $\log_{64} (x) = 1/3$

Evaluate without using your calculator.

27) $\ln \sqrt[3]{e}$	28) $\ln \frac{1}{e^2}$
29) $\ln e^{10}$	30) $\log_{10} (10^e)$

For problems 31 – 38: Evaluate using your calculator. Use the change of base formula if needed

31) log 20	32) ln 42
33) ln 2.9	34) log 0.5
35) $\log_4 36$	36) $\log_7 100$
37) $\log_{1.05} 3.5$	38) $\log_{1.067} 2$

5.4 Graphs and Properties of Logarithmic Functions

In this section, you will:

1. *examine properties of logarithmic functions*
2. *examine graphs of logarithmic functions*
3. *examine the relationship between graphs of exponential and logarithmic functions*

Recall that the exponential function $f(x) = 2^x$ produces this table of values

x	-3	-2	-1	0	1	2	3
f(x)	1/8	1/4	1/2	1	2	4	8

Since the logarithmic function is an inverse of the exponential, $g(x) = \log_2(x)$ produces the table of values

x	1/8	1/4	1/2	1	2	4	8
g(x)	-3	-2	-1	0	1	2	3

In this second table, notice that

1) As the input increases, the output increases.

2) As input increases, the output increases more slowly.

3) Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0, \infty)$.

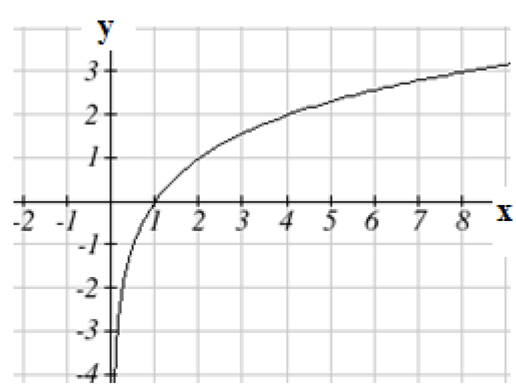
4) Since the exponential function can accept all real numbers as inputs, the logarithm can have any real number as output, so the range is all real numbers or $(-\infty, \infty)$.

Plotting the graph of $g(x) = \log_2(x)$ from the points in the table , notice that as the input values for x approach zero, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.

In symbolic notation we write

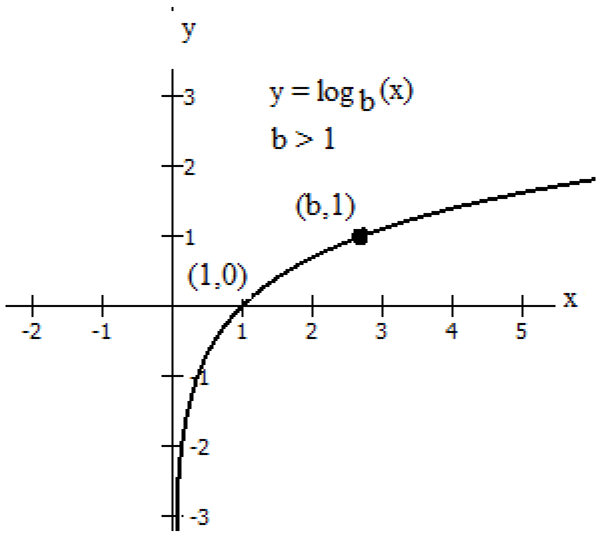
as $x \to 0^+$, $f(x) \to -\infty$

and as $x \to \infty$, $f(x) \to \infty$



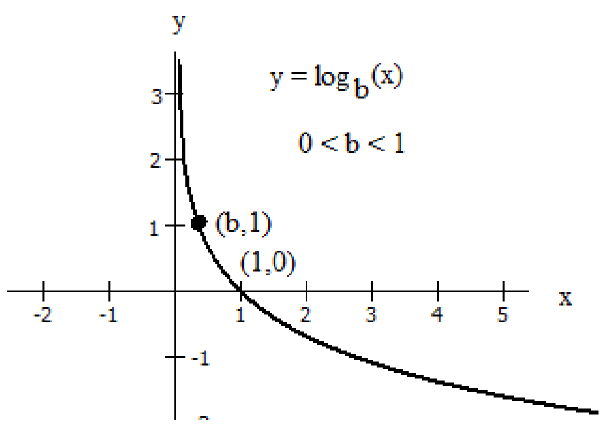
Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, "Chapter 4: Exponential and Logarithmic Functions," licensed under a Creative Commons CC BY-SA 3.0 license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

Graphically, in the function $g(x) = \log_b(x)$, b > 1, we observe the following properties:



- The graph has a horizontal intercept at (1, 0)
- The line x = 0 (the y-axis) is a vertical asymptote; as x $\rightarrow 0^+$, y$\rightarrow -\infty$
- The graph is increasing if b > 1
- The domain of the function is $x > 0$, or $(0, \infty)$
- The range of the function is all real numbers, or $(-\infty, \infty)$

However if the base b is less than 1, 0 < b < 1, then the graph appears as below. This follows from the log property of reciprocal bases : $\log_{1/b} C = -\log_b(C)$



- The graph has a horizontal intercept at (1, 0)
- The line x = 0 (the y-axis) is a vertical asymptote;; as x $\rightarrow 0^+$, y$\rightarrow \infty$
- The graph is decreasing if 0 < b < 1
- The domain of the function is $x > 0$, or $(0, \infty)$
- The range of the function is all real numbers, or $(-\infty, \infty)$

When graphing a logarithmic function, it can be helpful to remember that the graph will pass through the points (1, 0) and (*b*, 1).

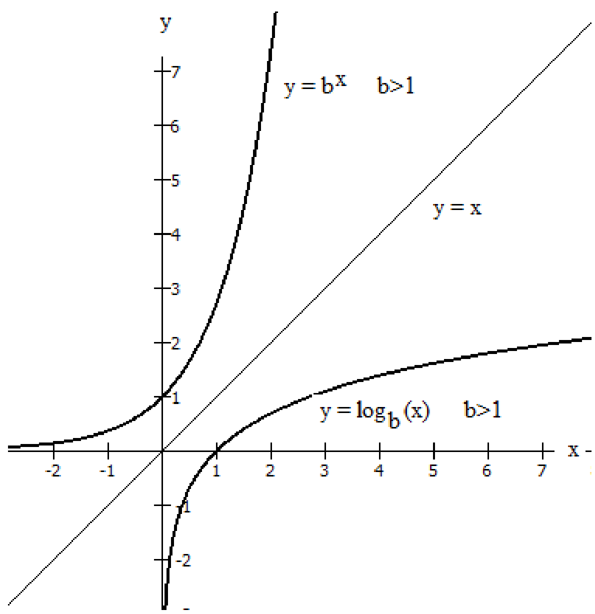
Finally, we compare the graphs of $y = b^x$ and $y = \log_b(x)$, shown below on the same axes.

Because the functions are inverse functions of each other, for every specific ordered pair (h, k) on the graph of $y = b^x$, we find the point (k, h) with the coordinates reversed on the graph of y = log b (x).

In other words, if the point with x = h and y = k is on the graph of $y = b^x$, then the point with x = k and y = h lies on the graph of $y = \log_b(x)$

The domain of $y = b^x$ is the range of $y = \log_b(x)$
The range of $y = b^x$ is the domain of $y = \log_b(x)$

For this reason, the graphs appear as reflections, or mirror images, of each other across the diagonal line y=x. This is a property of graphs of inverse functions that students should recall from their study of inverse functions in their prerequisite algebra class.



	$y = b^x$, with b>1	**$y = \log_b(x)$, with b>1**
Domain	all real numbers	all positive real numbers
Range	all positive real numbers	all real numbers
Intercepts	(0,1)	(1,0)
Asymptotes	Horizontal asymptote is the line y = 0 (the x-axis) As $x \to -\infty$, $y \to 0$	Vertical asymptote is the line x = 0 (the y axis) As $x \to 0^+$, $y \to -\infty$

SECTION 5.4 PROBLEM SET: GRAPHS AND PROPERIES OF LOGARITHMIC FUNCTIONS

Questions 1 – 3: For each of the following functions

a. Sketch a reasonably accurate graph showing the shape of the graph of the function
b. State the domain
c. State the range
d. State whether the graph has a vertical asymptote or a horizontal asymptote and write the equation of that asymptote
e. Does the graph have an x-intercept or a y-intercept asymptote? Write the coordinates of the x-intercept or the y-intercept.

1) $y = \ln x$ a. Sketch the graph below	b. domain:_______________ c. range:______________________ d. Is the asymptote horizontal or vertical? _____________________________ Equation of the asymptote:__________ e. Coordinates of x intercept or y intercept:_______
2) $y = \log x$ a. Sketch the graph below	b. domain:_______________ c. range:______________________ d. Is the asymptote horizontal or vertical? _____________________________ Equation of the asymptote:__________ e. Coordinates of x intercept or y intercept:______
3) $y = \log_{0.8} x$ a. Sketch the graph below	b. domain:_______________ c. range:______________________ d. Is the asymptote horizontal or vertical? _____________________________ Equation of the asymptote:__________ e. Coordinates of x intercept or y intercept:______

SECTION 5.4 PROBLEM SET: GRAPHS AND PROPERIES OF LOGARITHMIC FUNCTIONS

Questions 4 - 5: For the pair of inverse functions $y = e^x$ and $y = \ln x$

a. Sketch a reasonably accurate graph showing the shape of the graph of the function
b. State the domain
c. State the range
d. State whether the graph has a vertical asymptote or a horizontal asymptote and write the equation of that asymptote
e. Does the graph have an x-intercept or a y-intercept asymptote? Write the coordinates of the x-intercept or the y-intercept.

4) $y = e^x$ Sketch the graph below	5) $y = \ln x$ Sketch the graph below
b. domain:_______________ c. range:______________________ d. Is the asymptote horizontal or vertical? ______________________________ Equation of the asymptote:__________ e. Coordinates of x intercept or y intercept: _______	b. domain:_______________ c. range:______________________ d. Is the asymptote horizontal or vertical? ______________________________ Equation of the asymptote:_____________ e. Coordinates of x intercept or y intercept:_______

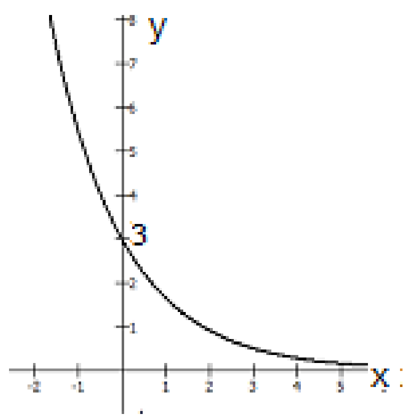
SECTION 5.4 PROBLEM SET: GRAPHS AND PROPERIES OF LOGARITHMIC FUNCTIONS

Questions 6-11: Match the graph with the function.
Choose the function from the list below and write it on the line underneath the graph.

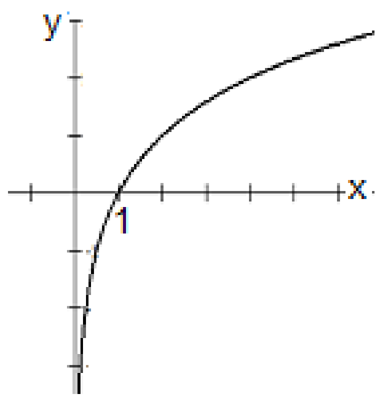
Hint: To match the function and the graph, identify these properties of the graph and function

- *Is the function increasing decreasing?*
- *Examine the asymptote*
- *Determine the x or y intercept*

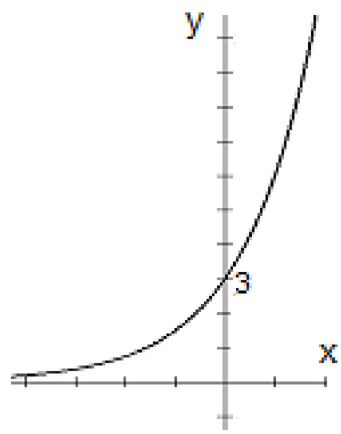
$y = 3(2^x)$ $y = 5(0.4^x)$ $y = \log_2(x)$ $y = \log_{1/2}(x)$ $y = 3e^{-0.6x}$ $y = 5e^{0.3x}$



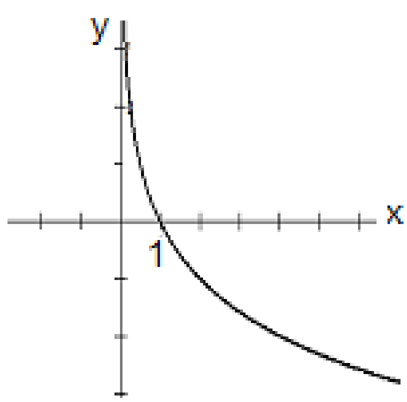
6) Function______________



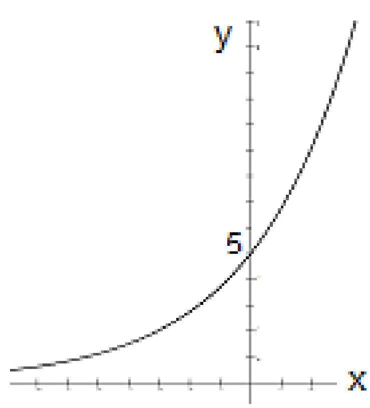
7) Function___________



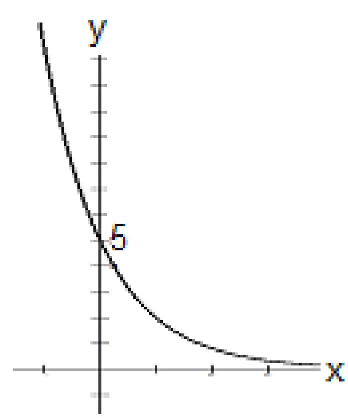
8) Function_________



9) Function______________



10) Function___________



11) Function ________

5.5 Application Problems with Exponential and Logarithmic Functions

In this section, you will:

1. *review strategies for solving equations arising from exponential formulas*
2. *solve application problems involving exponential functions and logarithmic functions*

STRATEGIES FOR SOLVING EQUATIONS THAT CONTAIN EXPONENTS

When solving application problems that involve exponential and logarithmic functions, we need to pay close attention to the position of the variable in the equation to determine the proper way solve the equation we investigate solving equations that contain exponents.

Suppose we have an equation in the form : value = coefficient(base) $^{\textbf{exponent}}$

We consider four strategies for solving the equation:

STRATEGY A: If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient(base) $^{\text{exponent}}$ to evaluate its value.

STRATEGY B: If the variable is the coefficient, evaluate the expression for (base) $^{\text{exponent}}$. Then it becomes a linear equation which we solve by dividing to isolate the variable.

STRATEGY C: If the variable is in the exponent, use logarithms to solve the equation.

STRATEGY D: If the variable is not in the exponent, but is in the base, use roots to solve the equation.

Below we examine each strategy with one or two examples of its use.

STRATEGY A: If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient(base) $^{\textbf{exponent}}$ **to evaluate its value.**

◆***Example 1***: Suppose that a stock's price is rising at the rate of 7% per year, and that it continues to increase at this rate. If the value of one share of this stock is \$43 now, find the value of one share of this stock three years from now.

Solution: Let y = the value of the stock after t years: $y = ab^t$

The problem tells us that a = 43 and r = 0.07, so b = 1+ r = 1+ 0.07 = 1.07

Therefore, function is $y = 43(1.07)^t$.

In this case we know that t = 3 years, and we need to evaluate y when t = 3.

At the end of 3 years, the value of this one share of this stock will be

$$y = 43(1.07)^3 = \$52.68$$

STRATEGY B: If the variable is the coefficient, evaluate the expression for $(\text{base})^{\text{exponent}}$. Then it becomes a linear equation which we solve by dividing to isolate the variable.

◆***Example 2:*** The value of a new car depreciates (decreases) after it is purchased. Suppose that the value of the car depreciates according to an exponential decay model. Suppose that the value of the car is \$12000 at the end of 5 years and that its value has been decreasing at the rate of 9% per year. Find the value of the car when it was new.

Solution: Let y be the value of the car after t years: $y = ab^t$

$r = -0.09$ and $b = 1+r = 1+(-0.09) = 0.91$

The function is $y = a(0.91)^t$

In this case we know that when t = 5, then y = 12000; substituting these values gives

$$12000 = a(0.91)^5$$

We need to solve for the initial value a, the purchase price of the car when new.

First evaluate $(0.91)^5$; then solve the resulting linear equation to find a.

$$12000 = a(0.624)$$

$$a = \frac{12000}{0.624} = \$19,230.77$$; The car's value was \$19,230.77 when it was new.

STRATEGY C: If the variable is in the exponent, use logarithms to solve the equation.

◆***Example 3***: A national park has a population of 5000 deer in the year 2016. Conservationists are concerned because the deer population is decreasing at the rate of 7% per year. If the population continues to decrease at this rate, how long will it take until the population is only 3000 deer?

Solution: Let y be the number of deer in the national park t years after the year 2016: $y = ab^t$

$r = -0.07$ and $b = 1+r = 1+(-0.07) = 0.93$ and the initial population is $a = 5000$

The exponential decay function is $y = 5000(0.93)^t$

To find when the population will be 3000, substitute y = 3000

$$3000 = 5000(0.93)^t$$

Next, divide both sides by 5000 to isolate the exponential expression

$$\frac{3000}{5000} = \frac{5000}{5000}(0.93)^t$$

$$0.6 = 0.93^t$$

Rewrite the equation in logarithmic form; then use the change of base formula to evaluate.

$$t = \log_{0.93}(0.6)$$

$$t = \frac{\ln(0.6)}{\ln(0.93)} = 7.039$$ years; After 7.039 years , there are 3000 deer.

Note: In Example 3, we needed to state the answer to several decimal places of precision to remain accurate. Evaluating the original function using a rounded value of t = 7 years gives a value that is close to 3000, but not exactly 3000.

$$y = 5000(0.93)^7 = 3008.5 \text{ deer}$$

However using t =7.039 years produces a value of 3000 for the population of deer

$$y = 5000(0.93)^{7.039} = 3000.0016 \approx 3000 \text{ deer}$$

◆***Example 4*** A video posted on YouTube initially had 80 views as soon as it was posted. The total number of views to date has been increasing exponentially according to the exponential growth function $y = 80e^{0.12t}$, where t represents time measured in days since the video was posted. How many days does it take until 2500 people have viewed this video?

Solution: Let y be the total number of views t days after the video is initially posted. We are given that the exponential growth function is $y = 80e^{0.12t}$ and we want to find the value of t for which y = 2500.
Substitute y = 2500 into the equation and use natural log to solve for t.

$$2500 = 80e^{0.12t}$$

Divide both sides by the coefficient, 80, to isolate the exponential expression.

$$\frac{2500}{80} = \frac{80}{80}e^{0.12t}$$

$$31.25 = e^{0.12t}$$

Rewrite the equation in logarithmic form

$$0.12t = \ln(31.25)$$

Divide both sides by 0.12 to isolate t; then use your calculator and its natural log function to evaluate the expression and solve for t.

$$t = \frac{\ln(31.25)}{0.12}$$

$$t = \frac{3.442}{0.12}$$

$$t \approx 28.7 \text{ days}$$

This video will have 2500 total views approximately 28.7 days after it was posted.

STRATEGY D: If the variable is not in the exponent, but is in the base, we use roots to solve the equation.
It is important to remember that we only use logarithms when the variable is in the exponent.

◆***Example 5*** A statistician creates a website to analyze sports statistics. His business plan states that his goal is to accumulate 50,000 followers by the end of 2 years (24 months from now). He hopes that if he achieves this goal his site will be purchased by a sports news outlet. The initial user base of people signed up as a result of pre-launch advertising is 400 people.
Find the monthly growth rate needed if the user base is to accumulate to 50,000 users at the end of 24 months.

Solution: Let y be the total user base t months after the site is launched.

The growth function for this site is $y = 400(1+r)^t$;

We don't know the growth rate r. We do know that when t = 24 months, then y = 50000.

Substitute the values of y and t; then we need to solve for r.

$$50000 = 400(1+r)^{24}$$

Divide both sides by 400 to isolate b24 on one side of the equation

$$\frac{50000}{400} = \frac{400}{400}(1+r)^{24}$$

$$125 = (1+r)^{24}$$

Because the variable in this equation is in the base, we use roots:

$$\sqrt[24]{125} = 1+r$$

$$125^{1/24} = 1+r$$

$$1.2228 \approx 1+r$$

$$0.2228 \approx r$$

The website's user base needs to increase at the rate of 22.28% per month in order to accumulate 50,000 users by the end of 24 months.

◆***Example 6*** A fact sheet on caffeine dependence from Johns Hopkins Medical Center states that the half life of caffeine in the body is between 4 and 6 hours. Assuming that the typical half life of caffeine in the body is 5 hours for the average person and that a typical cup of coffee has 120 mg of caffeine.
a. Write the decay function.
b. Find the hourly rate at which caffeine leaves the body.
c. How long does it take until only 20 mg of caffiene is still in the body?
https://www.hopkinsmedicine.org/psychiatry/research/bpru/docs/caffeine_dependence_fact_sheet.pdf

Solution: a. Let y be the total amount of caffeine in the body t hours after drinking the coffee.
Exponential decay function $y = ab^t$ models this situation.
The initial amount of caffeine is a = 120.

We don't know b or r, but we know that the half- life of caffeine in the body is 5 hours. This tells us that when t = 5, then there is half the initial amount of caffeine remaining in the body.

$$y = 120b^t$$

$$\frac{1}{2}(120) = 120b^5$$

$$60 = 120b^5$$

Divide both sides by 120 to isolate the expression b^5 that contains the variable.

$$\frac{60}{120} = \frac{120}{120}b^5$$

$$0.5 = b^5$$

The variable is in the base and the exponent is a number. Use roots to solve for b:

$$\sqrt[5]{0.5} = b$$

$$0.5^{1/5} = b$$

$$0.87 = b$$

We can now write the decay function for the amount of caffeine (in mg.) remaining in the body t hours after drinking a cup of coffee with 120 mg of caffeine

$$y = f(t) = 120\ (0.87)^{t}$$

b. Use b = 1 + r to find the decay rate r. Because b = 0.87 < 1 and the amount of caffeine in the body is decreasing over time, the value of r will be negative.

$$0.87 = 1 + r$$

$$r = -0.13$$

The decay rate is 13%; the amount of caffeine in the body decreases by 13% per hour.

c. To find the time at which only 20 mg of caffeine remains in the body, substitute y = 20 and solve for the corresponding value of t.

$$y = 120(.87)^t$$

$$20 = 120(.87)^t$$

Divide both sides by 120 to isolate the exponential expression.

$$\frac{20}{120} = \frac{120}{120}\left(.87^t\right)$$

$$0.1667 = .87^t$$

Rewrite the expression in logarithmic form and use the change of base formula

$$t = \log_{0.87}(0.1667)$$

$$t = \frac{\ln(0.1667)}{\ln(0.87)} \approx 12.9 \text{ hours}$$

After 12.9 hours, 20 mg of caffeine remains in the body.

EXPRESSING EXPONENTIAL FUNCTIONS IN THE FORMS $y = ab^t$ *and* $y = ae^{kt}$

Now that we've developed our equation solving skills, we revisit the question of expressing exponential functions equivalently in the forms $y = ab^t$ and $y = ae^{kt}$

We've already determined that if given the form $y = ae^{kt}$, it is straightforward to find b.

◆***Example 7*** For the following examples, assume t is measured in years.

a. Express $y = 3500\, e^{0.25t}$ in form $y = ab^t$ and find the annual percentage growth rate.

b. Express $y = 28000\, e^{-0.32t}$ in form $y = ab^t$ and find the annual percentage decay rate.

Solution: a. Express $y = 3500\, e^{0.25t}$ in the form $y = ab^t$

$y = ae^{kt} = ab^t$

$a(e^k)^t = ab^t$

Thus $e^k = b$

In this example $b = e^{0.25} \approx 1.284$

We rewrite the growth function as $y = 3500(1.284^t)$

To find r, recall that $b = 1+r$

$1.284 = 1+r$

$0.284 = r$

The continuous growth rate is $k = 0.25$ and the annual percentage growth rate is 28.4% per year.

b. Express $y = 28000\, e^{-0.32t}$ in the form $y = ab^t$

$y = ae^{kt} = ab^t$

$a(e^k)^t = ab^t$

Thus $e^k = b$

In this example $b = e^{-0.32} \approx 0.7261$

We rewrite the growth function as $y = 28000(0.7261^t)$

To find r, recall that $b = 1+r$

$0.7261 = 1+r$

$0.2739 = r$

The continuous decay rate is $k = -0.32$ and the annual percentage decay rate is 27.39% per year.

In the sentence, we omit the negative sign when stating the annual percentage decay rate because we have used the word "decay" to indicate that r is negative.

◆***Example 8*** a. Express $y = 4200\,(1.078)^t$ in the form $y = ae^{kt}$
b. Express $y = 150\,(0.73)^t$ in the form $y = ae^{kt}$

Solution: a. Express $y = 4200\,(1.078)^t$ in the form $y = ae^{kt}$

$y = ae^{kt} = ab^t$

$a(e^k)^t = ab^t$

$e^k = b$

$e^k = 1.078$

Therefore $k = \ln 1.078 \approx 0.0751$

We rewrite the growth function as $y = 3500e^{0.0751t}$

b. Express $y =150\,(0.73)^t$ in the form $y = ae^{kt}$

$y = ae^{kt} = ab^t$

$a(e^k)^t = ab^t$

$e^k = 0.73$

$e^k = 0.73$

Therefore $k = \ln 0.73 \approx -0.3147$

We rewrite the growth function as $y = 150\, e^{-0.3147t}$

AN APPLICATION OF A LOGARITHMIC FUNCTON

Suppose we invest $10,000 today and want to know how long it will take to accumulate to a specified amount, such as $15,000. The time t needed to reach a future value y is a logarithmic function of the future value: t = g(y)

◆***Example 9*** Suppose that Vinh invests $10000 in an investment earning 5% per year. He wants to know how long it would take his investment to accumulate to $12000, and how long it would take to accumulate to $15000.

Solution: We start by writing the exponential growth function that models the value of this investment as a function of the time since the $10000 is initially invested

$$y = 10000(1.05)^t$$

We divide both sides by 10000 to isolate the exponential expression on one side.

$$\frac{y}{10000} = 1.05^t$$

Next we rewrite this in logarithmic form to express time as a function of the accumulated future value. We'll use function notation and call this function g(y).

$$t = g(y) = \log_{1.05}\left(\frac{y}{10000}\right)$$

Use the change of base formula to express t as a function of y using natural logarithm:

$$t = g(y) = \frac{\ln\left(\frac{y}{10000}\right)}{\ln(1.05)}$$

We can now use this function to answer Vinh's questions.

To find the number of years until the value of this investment is \$12,000, we substitute y = \$12,000 into function g and evaluate t:

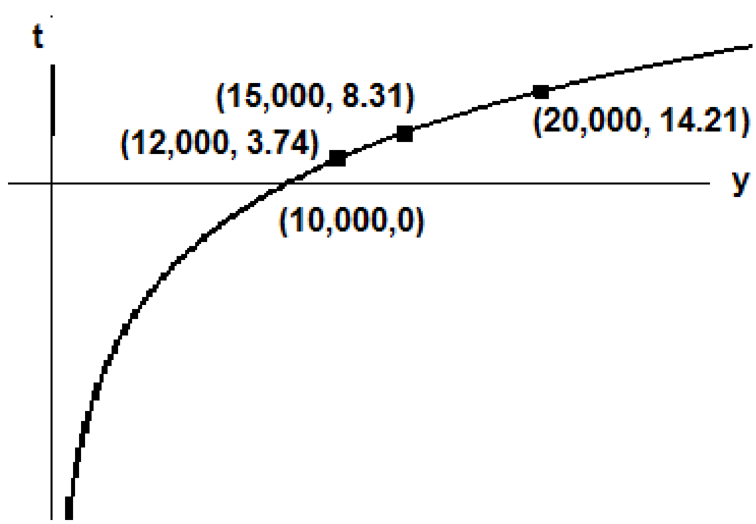
$$t = g(12000) = \frac{\ln\left(\frac{12000}{10000}\right)}{\ln(1.05)} = \frac{\ln(1.2)}{\ln(1.05)} = 3.74 \text{ years}$$

To find the number of years until the value of this investment is \$15,000, we substitute y = \$15,000 into function g and evaluate t:

$$t = g(15000) = \frac{\ln\left(\frac{15000}{10000}\right)}{\ln(1.05)} = \frac{\ln(1.5)}{\ln(1.05)} = 8.31 \text{ years}$$

Before ending this section, we investigate the graph of the function $t = g(y) = \frac{\ln\left(\frac{y}{10000}\right)}{\ln(1.05)}$.

We see that the function has the general shape of logarithmic functions that we examined in section 5.4. From the points plotted on the graph, we see that function g is an increasing function but it increases very slowly.

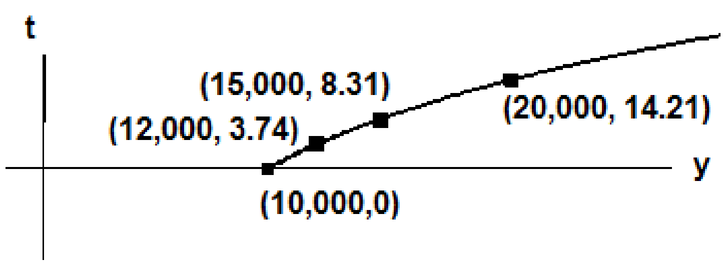


If we consider just the function $t = g(y) = \dfrac{\ln\left(\dfrac{y}{10000}\right)}{\ln(1.05)}$, then the domain of function would be y > 0, all positive real numbers, andthe range for t would be all real numbers.

In the context of this investment problem, the initial investment at time t = 0 is y =$10,000. Negative values for time do not make sense. Values of the investment that are lower than the initial amount of $10,000 also do not make sense for an investment that is increasing in value.

Therefore the function and graph as it pertains to this problem concerning investments has domain $y \geq 10{,}000$ and range $t \geq 0$.

The graph below is restricted to the domain and range that make practical sense for the investment in this problem.



SECTION 5.5 PROBLEM SET: APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1) An investment's value is rising at the rate of 5% per year. The initial value of the investment is $20,000 in 2016. a. Write the function that gives the value of the investment as a function of time t in years after 2016. b. Find the value of the investment in 2028 c. When will the value be $30,000?	2) The population of a city is increasing at the rate of 2.3% per year, since the year 2000. Its population in 2010 was 137,000 people. Find the population of the city in the year 2000.
3) The value of a piece of industrial equipment depreciates after it is purchased. Suppose that the depreciation follows an exponential decay model. The value of the equipment at the end of 8 years is $30,000 and its value has been decreasing at the rate of 7.5% per year. Find the initial value of the equipment when it was purchased.	4) An investment has been losing money. Its value has been decreasing at the rate of 3.2% per year. The initial value of the investment was $75,000 in 2010. a. Write the function that gives the value of the investment as a function of time t in years after 2010. b. If the investment's value continues to decrease at this rate, find the value of the investment in 2020.

SECTION 5.5 PROBLEM SET: APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5) A social media site has 275 members initially. The number of members has been increasing exponentially according to the function $y = 275e^{0.21t}$, where t is the number of months since the site's initial launch. How many months does it take until the site has 5000 members? *State answer to the nearest tenth of a month (1 decimal place).*

6) A city has a population of 62000 people in the year 2000. Due to high unemployment, the city's population has been decreasing at the rate of 2% per year. Using this model, find the population of this city in 2016.

7) A city has a population of 87,000 people in the year 2000. The city's population has been increasing at the rate of 1.5% per year. How many years does it take until the population reaches 100,000 people?

8) An investment of $50,000 is in increasing in value at the rate of 6.3% per year. How many years does it take until the investment is worth $70,000?

SECTION 5.5 PROBLEM SET: APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

9) A city has a population of 50,000 people in the year 2000. The city's population increases at a constant percentage rate. Fifteen years later, in 2015, the population of this city was 70,000. Find the annual percentage growth rate.	10) 200 mg of a medication is administered to a patient. After 3 hours, only 100 mg remains in the bloodstream. Using an exponential decay model, find the hourly decay rate.
11) An investment is losing money at a constant percentage rate per year. The investment was initially worth $25,000 but is worth only $20,000 after 4 years. Find the percentage rate at which the investment is losing value each year (that is, find the annual decay rate).	12) Using the information in question 11, how many years does it take until the investment is worth only half of its initial value?

SECTION 5.5 PROBLEM SET: APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

For question 13:

- if the function is given in the form $y = ae^{kt}$, rewrite it in the form $y = ab^t$.
- if the function is given in the form $y = ab^t$, rewrite it in the form $y = ae^{kt}$.

13a) $y = 7900e^{0.472t}$. Write in the form $y = ab^t$	13b) $y = 4567(0.67^t)$ Write in the form $y = ae^{kt}$
13c) $y = 18720(1.47^t)$ Write in the form $y = ae^{kt}$	13d) $y = 1200e^{-0.078t}$. Write in the form $y = ab^t$

SECTION 5.6 PROBLEM SET: CHAPTER REVIEW

1) The value of a new boat depreciates after it is purchased. The value of the boat 7 years after it was purchased is $25,000 and its value has been decreasing at the rate of 8.2% per year.

   a. Find the initial value of the boat when it was purchased.

   b. How many years after it was purchased will the boat's value be $20,000?

   c. What was its value 3 years after the boat was purchased?

2) Tony invested $40,000 in 2010; unfortunately his investment has been losing value at the rate of 2.7% per year.

   a. Write the function that gives the value of the investment as a function of time t in years after 2010.

   b. Find the value of the investment in 2020, if its value continues to decrease at this rate.

   c. In what year will the investment be worth half its original value?

3) Rosa invested $25,000 in 2005; its value has been increasing at the rate of 6.4% annually.

   a. Write the function that gives the value of the investment as a function of time t in years after 2005.

   b. Find the value of the investment in 2025.

4) The population of a city is increasing at the rate of 3.2% per year, since the year 2000. Its population in 2015 was 235,000 people.

   a. Find the population of the city in the year 2000.

   b. In what year with the population be 250, 000 if it continues to grow at this rate.

   c. What was the population of this city in the year 2008?

5) The population of an endangered species has only 5000 animals now. Its population has been decreasing at the rate of 12% per year.

   a. If the population continues to decrease at this rate, how many animals will be in this population 4 years from now.

   b. In what year will there be only 2000 animals remaining in this population?

6) 300 mg of a medication is administered to a patient. After 5 hours, only 80 mg remains in the bloodstream.

   a. Using an exponential decay model, find the hourly decay rate.

   b. How many hours after the 300 mg dose of medication was administered was there 125 mg in the bloodstream

   c. How much medication remains in the bloodstream after 8 hours?

7) If $y = 240b^t$ and y = 600 when t = 6 years, find the annual growth rate. State your answer as a percent.

8) If the function is given in the form $y = ae^{kt}$, rewrite it in the form $y = ab^t$.
   If the function is given in the form $y = ab^t$, rewrite it in the form $y = ae^{kt}$.

   a. $y = 375000(1.125^t)$ b. $y = 5400e^{0.127t}$ c. $y = 230e^{-0.62t}$ d. $y = 3600(0.42^t)$

CHAPTER 6

Chapter 6: Mathematics Of Finance

In this chapter, you will learn to:

1. *Solve financial problems that involve simple interest.*
2. *Solve problems involving compound interest.*
3. *Find the future value of an annuity, and the amount of payments to a sinking fund.*
4. *Find the future value of an annuity, and an installment payment on a loan.*

6.1 Simple Interest and Discount

In this section, you will learn to:

1. *Find simple interest.*
2. *Find present value.*
3. *Find discounts and proceeds.*

SIMPLE INTEREST

It costs to borrow money. The rent one pays for the use of money is called the **interest.** The amount of money that is being borrowed or loaned is called the **principal** or **present value**. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principal for the time period he keeps the money. Although the interest rate is often specified for a year, it may be specified for a week, a month, or a quarter, etc. The credit card companies often list their charges as monthly rates, sometimes it is as high as 1.5% a month.

SIMPLE INTEREST

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$\mathbf{I = P \cdot r \cdot t}$$

The total amount A, also called the accumulated value or the future value, is given by

$$A = P + I = P + Prt$$

or $$\mathbf{A = P(1 + rt)}$$

where interest rate r is expressed in decimals.

◆***Example 1*** Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

Solution: The interest is computed by multiplying the principal with the interest rate and the time.

$$I = Prt$$

$$I = \$600(.15)\frac{5}{12} \quad = \$37.50$$

The total amount is

$$A = P + I = \$600 + \$37.50 \quad = \$637.50$$

Incidentally, the total amount can be computed directly as

$$A = P(1 + rt) \quad = \$600[1 + (.15)(5/12)]$$
$$= \$600(1 + .0625)$$
$$= \$637.50$$

◆***Example 2*** Jose deposited \$2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

Solution: The total amount or the future value is given by A = P(1 + rt).

$$A = \$2500[1 + (.06)(3)]$$
$$A \quad = \$2950$$

◆***Example 3*** Darnel owes a total of \$3060 which includes 12% interest for the three years he borrowed the money. How much did he originally borrow?

Solution: This time we are asked to compute the principal P.

$$\$3060 = P[1 + (.12)(3)]$$
$$\$3060 \ = P(1.36)$$
$$\frac{\$3060}{1.36} \ = P$$
$$\$2250 \ = P$$

Darnel originally borrowed \$2250.

◆***Example 4*** A Visa credit card company charges a 1.5% finance charge each month on the unpaid balance. If Martha owed \$2350 and has not paid her bill for three months, how much does she owe now?

Solution: Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$A = \$2350 + \$2350(.015)(3) = \ \$2350 + \$105.75 \quad = \$2455.75$$

Alternatively, again, we can compute the amount directly by using formula A = P(1 + rt)

$$A = \$2350[1 + (.015)(3)] \ = \$2350(1.045) \quad = \$2455.75$$

DISCOUNTS AND PROCEEDS

Banks often deduct the simple interest from the loan amount at the time that the loan is made. When this happens, we say the loan has been **discounted.** The interest that is deducted is called the **discount**, and the actual amount that is given to the borrower is called the **proceeds**. The amount the borrower is obligated to repay is called the **maturity value.**

DISCOUNT AND PROCEEDS

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$\mathbf{D = M \cdot r \cdot t}$$

The proceeds P, the actual amount the borrower gets, is given by

$$P = M - D$$

$$P = M - Mrt$$

or $$\mathbf{P = M(1 - rt)}$$

where interest rate r is expressed in decimals.

◆***Example 5*** Francisco borrows \$1200 for 10 months at a simple interest rate of 15% per year. Determine the discount and the proceeds.

Solution: The discount D is the interest on the loan that the bank deducts from the loan amount.

$$D = Mrt$$

$$D = \$1200(.15)\left(\frac{10}{12}\right) = \$150$$

Therefore, the bank deducts \$150 from the maturity value of \$1200, and gives Francisco \$1050. Francisco is obligated to repay the bank \$1200.

In this case, the discount D = \$150, and the proceeds P = \$1200 – \$150 = \$1050.

◆***Example 6*** If Francisco wants to receive \$1200 for 10 months at a simple interest rate of 15% per year, what amount of loan should he apply for?

Solution: In this problem, we are given the proceeds P and are being asked to find the maturity value M.

We have P = \$1200, r = .15, t = 10/12 . We need to find M.

We know $P = M - D$

but also $D = Mrt$

therefore $P = M - Mrt = M(1 - rt)$

$$\$1200 = M\left[1 - (.15)\left(\frac{10}{12}\right)\right]$$

We need to solve for M.

$\$1200 = M(1 - .125)$

$\$1200 = M(.875)$

$\frac{\$1200}{.875} = M$

$\$1371.43 = M$

Therefore, Francisco should ask for a loan for $1371.43.

The bank will discount $171.43 and Francisco will receive $1200.

SECTION 6.1 SUMMARY

Below is a summary of the formulas we developed for calculations involving simple interest:

SIMPLE INTEREST

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$\mathbf{I = P \cdot r \cdot t}$$

The total amount A also called the accumulated value or the future value is given by

$$A = P + I = P + Prt$$

or $$A = P(1 + rt)$$

where the interest rate r is expressed in decimals.

DISCOUNT AND PROCEEDS

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower receives at the time the money is borrowed, is given by

$$P = M - D$$

$$P = M - Mrt$$

or $$P = M(1 - rt)$$

where interest rate r is expressed in decimals.

At the end of the loan’s term, the borrower repays the entire maturity amount M.

SECTION 6.1 PROBLEM SET: SIMPLE INTEREST AND DISCOUNT

Do the following simple interest problems.

1) If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest?	2) You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest?
3) John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years?	4) Jessica takes a loan of $800 for 4 months at 12% simple interest. How much does she owe at the end of the 4-month period?
5) If an amount of $2,160, which includes a 10% simple interest for 2 years, is paid back, how much was borrowed 2 years earlier?	6) Jamie just paid off a loan of $2,544, the principal and simple interest. If he took out the loan six months ago at 12% simple interest, what was the amount borrowed?
7) Shanti charged $800 on her charge card and did not make a payment for six months. If there is a monthly charge of 1.5%, how much does she owe?	8) A credit card company charges 18% interest on the unpaid balance. If you owed $2000 three months ago and have been delinquent since, how much do you owe?

SECTION 6.1 PROBLEM SET: SIMPLE INTEREST AND DISCOUNT

Do the following simple interest problems.

9) An amount of \$2000 is borrowed for 3 years. At the end of the three years, \$2660 is paid back. What was the simple interest rate?	10) Nancy borrowed \$1,800 and paid back \$1,920, four months later. What was the simple interest rate?
11) Jose agrees to pay \$2,000 in one year at an interest rate of 12%. The bank subtracts the discount of 12% of \$2,000, and gives the rest to Jose. Find the amount of the discount and the proceeds to Jose.	12) Tasha signs a note for a discounted loan agreeing to pay \$1200 in 8 months at an 18% discount rate. Determine the amount of the discount and the proceeds to her.
13) An amount of \$8,000 is borrowed at a discount rate of 12%, find the proceeds if the length of the loan is 7 months.	14) An amount of \$4,000 is borrowed at a discount rate of 10%, find the proceeds if the length of the loan is 180 days.
15) Derek needs \$2400 new equipment for his shop. He can borrow this money at a discount rate of 14% for a year. Find the amount of the loan he should ask for so that his proceeds are \$2400.	16) Mary owes Jim \$750, and wants to repay him. Mary decides to borrow the amount from her bank at a discount rate of 16%. If she borrows the money for 10 months, find the amount of the loan she should ask for so that her proceeds are \$750?

6.2 Compound Interest

In this section you will learn to:

1. *Find the future value of a lump-sum.*
2. *Find the present value of a lump-sum.*
3. *Find the effective interest rate.*

COMPOUND INTEREST

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principal, but also on the past interest, then we say the interest is **compounded.**

Suppose we deposit \$200 in an account that pays 8% interest. At the end of one year, we will have \$200 + \$200(.08) = \$200(1 + .08) = \$216.

Now suppose we put this amount, \$216, in the same account. After another year, we will have \$216 + \$216(.08) = \$216(1 + .08) = \$233.28.

So an initial deposit of \$200 has accumulated to \$233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only \$232. The reason the amount is slightly higher is because the interest (\$16) we earned the first year, was put back into the account. And this \$16 amount itself earned for one year an interest of \$16(.08) = \$1.28, thus resulting in the increase. So we have earned interest on the principal as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, \$233.28, in the bank for another year, the final amount will be \$233.28 + \$233.28(.08) = \$233.28(1 + .08) = \$251.94.

Now let us look at the mathematical part of this problem so that we can devise an easier way to solve these problems.

After one year, we had \$200(1 + .08) = \$216

After two years, we had \$216(1 + .08)

But \$216 = \$200(1 + .08), therefore, the above expression becomes

$$\$200(1+.08)(1+.08) = \$200(1+.08)^2 = \$233.28$$

After three years, we get

$$\$233.28(1+.08) = \$200(1+.08)(1+.08)(1+.08)$$

which can be written as

$$\$200(1 + .08)^3 = \$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1 + .08)^5 = \$293.87$$

We summarize as follows:

The original amount	\$200	= \$200
The amount after one year	\$200(1 + .08)	= \$216
The amount after two years	$\$200(1 + .08)^2$	= \$233.28
The amount after three years	$\$200(1 + .08)^3$	= \$251.94
The amount after five years	$\$200(1 + .08)^5$	= \$293.87
The amount after t years	$\$200(1 + .08)^t$	

COMPOUNDING PERIODS

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit \$200 in the bank, after one quarter we will have $\$200(1 + \frac{.08}{4})$ or \$204.

After two quarters, we will have $\$200(1 + \frac{.08}{4})^2$ or \$208.08.

After one year, we will have $\$200(1 + \frac{.08}{4})^4$ or \$216.49.

After three years, we will have $\$200(1 + \frac{.08}{4})^{12}$ or \$253.65, etc.

The original amount	\$200	= \$200
The amount after one quarter	$\$200(1 + \frac{.08}{4})$	= \$204
The amount after two quarters	$\$200(1 + \frac{.08}{4})^2$	= \$208.08
The amount after one year	$\$200(1 + \frac{.08}{4})^4$	= \$216.49
The amount after two years	$\$200(1 + \frac{.08}{4})^8$	= \$234.31
The amount after three years	$\$200(1 + \frac{.08}{4})^{12}$	= \$253.65
The amount after five years	$\$200(1 + \frac{.08}{4})^{20}$	= \$297.19
The amount after t years	$\$200(1 + \frac{.08}{4})^{4t}$	

Therefore, if we invest a lump-sum amount of P dollars at an interest rate r, compounded n times a year, then after t years the final amount is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

The following examples use the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$

◆***Example 1*** If \$3500 is invested at 9% compounded monthly, what will the future value be in four years?

Solution: Clearly an interest of .09/12 is paid every month for four years. The interest is compounded 4×12 = 48 times over the four-year period. We get

$$A = \$3500\left(1+\frac{.09}{12}\right)^{48} = \$3500(1.0075)^{48} = \$5009.92$$

\$3500 invested at 9% compounded monthly will accumulate to \$5009.92 in four years.

◆***Example 2*** How much should be invested in an account paying 9% compounded daily for it to accumulate to \$5,000 in five years?

Solution: We know the future value, but need to find the principal.

$$\$5000 = P\left(1+\frac{.09}{365}\right)^{365\times 5}$$

$$\$5000 = P\,(1.568225)$$

$$\$3188.32 = P$$

\$3188,32 invested into an account paying 9% compounded daily will accumulate to \$5,000 in five years.

◆***Example 3*** If \$4,000 is invested at 4% compounded annually, how long will it take to accumulate to \$6,000?

Solution: n = 1 because annual compounding means compounding only once per year.
The formula simplifies to $A = (1+r)^t$ when n = 1.

$$\$6000 = 4000(1+.04)^t$$

$$\frac{6000}{4000} = 1.04^t$$

$$1.5 = 1.04^t$$

We use logarithms to solve for the value of t because the variable t is in the exponent.

$$t = \log_{1.04}(1.5)$$

Using the change of base formula we can solve for t:

$$t = \frac{\ln(1.5)}{\ln(1.04)} = 10.33 \text{ years}$$

It takes 10.33 years for \$4000 to accumulate to \$6000 if invested at 4% interest, compounded annually

◆***Example 4*** If \$5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of \$8000.

Solution: We have n = 4 for quarterly compounding.

$$\$8000 = \$5000\left(1+\frac{r}{4}\right)^{4\times 6}$$

$$\frac{\$8000}{\$5000} = \left(1+\frac{r}{4}\right)^{24}$$

$$1.6 = \left(1+\frac{r}{4}\right)^{24}$$

We use roots to solve for t because the variable r is in the base, whereas the exponent is a known number.

$$\sqrt[24]{1.6} = 1+\frac{r}{4}$$

Many calculators have a built in "nth root" key or function. In the TI-84 calculator, this is found in the Math menu. Roots can also be calculated as fractional exponents; if necessary, the previous step can be rewritten as

$$1.6^{1/24} = 1+\frac{r}{4}$$

Evaluating the left side of the equation gives

$$1.0197765 = 1+\frac{r}{4}$$

$$0.0197765 = \frac{r}{4}$$

$$r = 4(0.0197765) = 0.0791$$

An interest rate of 7.91% is needed in order for \$5000 invested now to accumulate to \$8000 at the end of 6 years, with interest compounded quarterly.

EFFECTIVE INTEREST RATE:

Banks are required to state their interest rate in terms of an **"effective yield"** or **"effective interest rate"**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. The next example shows how to calculate the effective rate.

To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example 5 illustrates how to calculate the effective rate.

◆***Example 5*** If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate? If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate? Which bank pays more interest?

Solution: Bank A: Suppose we deposit $1 in this bank and leave it for a year, we will get

$$1\left(1+\frac{0.072}{12}\right)^{12}=1.0744$$

$$r_{EFF} = 1.0744 - 1 = 0.0744$$

We earned interest of \$1.0744 – \$1.00 = \$.0744 on an investment of \$1.

The effective interest rate is 7.44%, often referred to as the APY or APR.

Bank B: The effective rate is calculated as

$$r_{EFF} = 1\left(1+\frac{0.072}{2}\right)^{2}-1=.0738$$

The effective interest rate is 7.38%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

CONTINUOUS COMPOUNDING

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is **compounded continuously**. Our next objective is to derive a formula to model continuous compounding.

Suppose we put \$1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be \$1(1 + 1) = \$2.

If the interest is compounded semiannually, in one year we will have $\$1(1 + 1/2)^2 = \$2.25$

If the interest is compounded quarterly, in one year we will have $\$1(1 + 1/4)^4 = \$2.44$

If the interest is compounded monthly, in one year we will have $\$1(1 + 1/12)^{12} = \$2.61$

If the interest is compounded daily, in one year we will have $\$1(1 + 1/365)^{365} = \$2.71$

We show the results as follows:

Frequency of compounding	Formula	Total amount
Annually	$1(1 + 1)	$2
Semiannually	$\$1(1 + 1/2)^2$	$2.25
Quarterly	$\$1(1 + 1/4)^4$	$2.44140625
Monthly	$\$1(1 + 1/12)^{12}$	$2.61303529
Daily	$\$1(1 + 1/365)^{365}$	$2.71456748
Hourly	$\$1(1 + 1/8760)^{8760}$	$2.71812699
Every minute	$\$1(1 + 1/525600)^{525600}$	$2.71827922
Every Second	$\$1(1 + 1/31536000)^{31536000}$	$2.71828247
Continuously	$1(2.718281828...)	$2.718281828...

We have noticed that the $1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "*e*" after the great mathematician Euler.

In mathematics, we say that as n becomes infinitely large the expression $\left(1+\frac{1}{n}\right)^n$ equals *e*.

Therefore, it is natural that the number e play a part in continuous compounding.

It can be shown that as n becomes infinitely large the expression $\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$

Therefore, it follows that if we invest $P at an interest rate r per year, compounded continuously, after t years the final amount will be given by

$$A = P \cdot e^{rt}.$$

◆***Example 6*** $3500 is invested at 9% compounded continuously. Find the future value in 4years.

Solution: Using the formula for the continuous compounding, we get $A = Pe^{rt}$.

$$A = \$3500e^{.09\times 4}$$

$$A = \$3500e^{.36}$$

$$A = \$5016.65$$

◆***Example 7*** If an amount is invested at 7% compounded continuously, what is the effective interest rate?

Solution: If we deposit $1 in the bank at 7% compounded continuously for one year, and subtract that $1 from the final amount, we get the effective interest rate in decimals.

$$r_{EFF} = 1e^{.07} - 1$$

$$r_{EFF} = 1.0725 - 1$$

$$r_{EFF} = .0725 \text{ or } 7.25\%$$

◆***Example 8*** If an amount is invested at 7% compounded continuously, how long will it take to double?

We offer two solutions.

Solution 1 uses logarithms to calculate the exact answer, so it is preferred. We already used this method in Example 3 to solve for time needed for an investment to accumulate to a specified future value.

Solution 2 provides an estimated solution that is applicable only to doubling time, but not to other multiples. Students should find out from their instructor if there is a preference as to which solution method is to be used for doubling time problems.

Solution: ***Solution 1:*** **Calculating the answer exactly:** $Pe^{.07t} = A$.

We don't know the initial value of the prinicipal but we do know that the accumulated value is double (twice) the principal.

$$Pe^{.07t} = 2P$$

We divide both sides by P

$$e^{.07t} = 2$$

Using natural logarithm:

$$.07t = \ln(2)$$

$$t = \ln(2)/.07 = 9.9 \text{ years}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

Solution 2: **Estimating the answer using the Law of 70:**

The Law of 70 is a useful tool for estimating the time needed for an investment to double in value. It is an approximation and is not exact and comes from our previous solution. We calculated that

$t = \ln(2)/ r$ where r was 0.07 in that solution

Evaluating $\ln(2) = 0.693$, gives $t = 0.693/r$. Multiplying numerator and denominator by 100 gives $t = 69.3/ (100r)$

If we estimate 69.3 by 70 and state the interest rate as a percent instead of a decimal, we obtain the Law of 70:

Law of 70: The number of years required to double money ≈ 70 ÷ interest rate

- Note that this is an approximate estimate only.
- The interest rate is stated as a percent (not decimal) in the Law of 70.

Using the Law of 70 gives us $t \approx 70/7=10$ which is close to but not exactly the value of 9..9 years calculated in Solution 1.

Approximate Doubling Time in Years as a Function of Interest Rate

Annual interest rate	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
Number of years to double money	70	35	23	18	14	12	10	9	8	7

The pattern in the table approximates the Law of 70.

With technology available to do calculations using logarithms, we would use the Law of 70 only for quick estimates of doubling times. Using the Law of 70 as an estimate works only for doubling times, but not other multiples, so it's not a replacement for knowing how to find exact solutions.

However, the Law of 70 can be useful to help quickly estimate many "doubling time" problems mentally, which can be useful in compound interest applications as well as other applications involving exponential growth.

◆***Example 9*** a. At the peak growth rate in the 1960's the worlds population had a doubling time of 35 years. At that time, approximately what was the growth rate?

b. As of 2015, the world population's annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

Solution: a. According to the law of 70,

doubling time = 35 ≈ 70 ÷ r

r ≈ 2 expressed as a percent

Therefore, the world population was growing at an approximate rate of 2% in the 1960's.

b.. According to the law of 70,

doubling time t ≈ 70 ÷ r = 70 ÷ 1.14 ≈ 61 years

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approcimately 61 years for the population to double.

SECTION 6.2 SUMMARY

Below is a summary of the formulas we developed for calculations involving compound interest:

COMPOUND INTEREST n times per year

1. If an amount P is invested for t years at an interest rate r per year, compounded n times a year, then the future value is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

P is called the principal and is also called the present value.

2. If a bank pays an interest rate r per year, compounded n times a year, then the effective interest rate is given by

$$\mathbf{r_{EFF}} = \left(1+\frac{r}{n}\right)^{n} - 1$$

CONTINUOUSLY COMPOUNDED INTEREST

3. If an amount P is invested for t years at an interest rate r per year, compounded continuously, then the future value is given by

$$\mathbf{A = P}e^{\mathbf{rt}}$$

4. If a bank pays an interest rate r per year, compounded n times a year, then the effective interest rate is given by

$$\mathbf{r_{EFF}} = e^{\mathbf{r}} - \mathbf{1}$$

5. The Law of 70 states that

The number of years to double money is approximately 70 ÷ interest rate

SECTION 6.2 PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems involving a lump-sum amount.

1) What will the final amount be in 4 years if \$8,000 is invested at 9.2% compounded monthly.?	2) How much should be invested at 10.3% for it to amount to \$10,000 in 6 years?
3) Lydia's aunt Rose left her \$5,000. Lydia spent \$1,000 on her wardrobe and deposited the rest in an account that pays 6.9% compounded daily. How much money will she have in 5 years?	4) Thuy needs \$1,850 in eight months for her college tuition. How much money should she deposit lump sum in an account paying 8.2% compounded monthly to achieve that goal?
5) Bank A pays 5% compounded daily, while Bank B pays 5.12% compounded monthly. Which bank pays more? Explain.	6) EZ Photo Company needs five copying machines in 2 1/2 years for a total cost of \$15,000. How much money should be deposited now to pay for these machines, if the interest rate is 8% compounded semiannually?
7) Jon's grandfather was planning to give him \$12,000 in 10 years. Jon has convinced his grandfather to pay him \$6,000 now, instead. If Jon invests this \$6,000 at 7.5% compounded continuously, how much money will he have in 10 years?	8) What will be the price of a \$20,000 car in 5 years if the inflation rate is 6%?

SECTION 6.2 PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems.

9) At an interest rate of 8% compounded continuously, how many years will it take to double your money?	10) If an investment earns 10% compounded continuously, in how many years will it triple? .
11) The City Library ordered a new computer system costing $158,000; it will be delivered in 6 months. The full amount will be due 30 days after delivery. How much must be deposited today into an account paying 7.5% compounded monthly to have $158,000 in 7 months?	12) Mr. and Mrs. Tran are expecting a baby girl in a few days. They want to put away money for her college education now. How much money should they deposit in an account paying 10.2% so they will have $100,000 in 18 years to pay for their daughter's educational expenses?
13) Find the effective interest rate for an account paying 7.2% compounded quarterly.	14) If a bank pays 5.75% compounded monthly, what is the effective interest rate?
15) The population of the African nation of Cameroon was 12 million people in the year 2015; it has been growing at the rate of 2.5% per year. If the population continues to grow that rate, what will the population be in 2030? (http://databank.worldbank.org/data on 4/26/2016)	16) According to the Law of 70, if an amount grows at an annual rate of 1%, then it doubles every seventy years. Suppose a bank pays 5% interest, how long will it take for you to double your money? How about at 15%?

6.3 Annuities and Sinking Funds

In this section, you will learn to:

1. *Find the future value of an annuity.*
2. *Find the amount of payments to a sinking fund.*

ORDINARY ANNUITY

In the first two sections of this chapter, we examined problems where an amount of money was deposited lump sum in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an **annuity**. And this is the subject of this section.

To develop a formula to find the value of an annuity, we will need to recall the formula for the sum of a geometric series.

A geometric series is of the form: $a + ax + ax^2 + ax^3 + \ldots + ax^n$.

In a geometric series, each subsequent term is obtained by multiplying the preceding term by a number, called the common ratio. A geometric series is completely determined by knowing its first term, the common ratio, and the number of terms.

In the example, $a + ax + ax^2 + ax^3 + \ldots + ax^{n-1}$ the first term of the series is a, the common ratio is x, and the number of terms is n.

The following are some examples of geometric series.

3 + 6 + 12 + 24 + 48 has first term a = 3 and common ratio x = 2

2 + 6 + 18 + 54 + 162 has first term a = 2 and common ratio x = 3

37 + 3.7 + .37 + .037 + .0037 has first term a = 35 and common ratio x = 0.1

In your algebra class, you developed a formula for finding the sum of a geometric series. You probably used r as the symbol for the ratio, but we are using x because r is the symbol we have been using for the interest rate.

The formula for the sum of a geometric series with first term *a* and common ratio *x* is:

$$\frac{a(x^n - 1)}{x - 1}$$

We will use this formula to find the value of an annuity.

Consider the following example.

◆***Example 1*** If at the end of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution: There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on.

The first payment of $500 will accumulate to an amount of $\$500(1 + .08/12)^{59}$.

The second payment of $500 will accumulate to an amount of $\$500(1 + .08/12)^{58}$.

The third payment will accumulate to $\$500(1 + .08/12)^{57}$.

The fourth payment will accumulate to $\$500(1 + .08/12)^{56}$.

And so on . . .

Finally the next to last (59th) payment will accumulate to $\$500(1 + .08/12)^{1}$.

The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we need to find the sum of the following series.

$$\$500(1 + .08/12)^{59} + \$500(1 + .08/12)^{58} + \$500(1 + .08/12)^{57} + \ldots + \$500$$

Written backwards, we have

$$\$500 + \$500(1 + .08/12) + \$500(1 + .08/12)^{2} + \ldots + \$500(1 + .08/12)^{59}$$

This is a geometric series with a = \$500, r = (1 + .08/12), and n = 59. The sum is

$$\frac{\$500[(1 + .08/12)^{60} - 1]}{.08/12}$$

$$= \$500(73.47686)$$

$$= \$36738.43$$

When the payments are made at the end of each period rather than at the beginning, we call it an **ordinary annuity.**

Future Value of an Ordinary Annuity

If a payment of m dollars is made in an account n times a year at an interest r, then the final amount A after t years is

$$\mathbf{A = \frac{m[(1 + r/n)^{nt} - 1]}{r/n}}$$

The future value is also called the accumulated value

◆***Example 2*** Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

Solution: The future value of this annuity can be found using the above formula.

$$A = \frac{\$300[(1 + .0575/4)^{16} - 1]}{.0575/4}$$

$$A = \$300(17.8463) = \$5353.89$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for 4 years, then at the end of 4 years she will have \$5,353.89

◆***Example 3*** Robert needs $5,000 in three years. How much should he deposit each month in an account that pays 8% compounded monthly in order to achieve his goal?

Solution: If Robert saves m dollars per month, after three years he will have

$$\frac{m[(1+.08/12)^{36}-1]}{.08/12}$$

But we'd like this amount to be $5,000. Therefore,

$$\frac{m[(1+.08/12)^{36}-1]}{.08/12} = \$5000$$

$$m\,(40.5356) = \$5000$$

$$m = \frac{5000}{40.5356} = \$123.35$$

Robert needs to deposit $123.35 at the end of each month for 3 years into an account paying 8% compounded monthly in order to have $5,000 at the end of 5 years.

SINKING FUND

When a business deposits money at regular intervals into an account in order to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**". Calculating the sinking fund deposit uses the same method as the previous problem.

◆***Example 4*** A business needs $450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

Solution: Again, suppose that m dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be $450,000. This suggests the following relationship:

$$\frac{m\,[(1+.09/4)^{20}-1]}{.09/4} = \$450{,}000$$

$$m\,(24.9115) = 450{,}000$$

$$m = \frac{450000}{24.9115} = \$\ 18{,}063.93$$

The business needs to deposit $18063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have $450,000 at the end of 5 years.

ANNUITY DUE

If the payment is made at the beginning of each period, rather than at the end, we call it an **annuity due**. The formula for the annuity due can be derived in a similar manner. Reconsider Example 1, with the change that the deposits are made at the beginning of each month.

◆***Example 5*** If at the beginning of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution: There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.

The first payment of $500 will accumulate to an amount of $\$500(1 + .08/12)^{60}$.

The second payment of $500 will accumulate to an amount of $\$500(1 + .08/12)^{59}$.

The third payment will accumulate to $\$500(1 + .08/12)^{58}$.

And so on . . .

The last payment is in the account for a month and accumulates to $500(1 + .08/12)

To find the total amount in five years, we need to find the sum of the series:

$$\$500(1 + .08/12)^{60} + \$500(1 + .08/12)^{59} + \$500(1 + .08/12)^{58} + \ldots + \$500(1 + .08/12)$$

Written backwards, we have

$$\$500(1 + .08/12) + \$500(1 + .08/12)^{2} + \ldots + \$500(1 + .08/12)^{60}$$

If we add $500 to this series, and later subtract that $500, the value will not change. We get

$$\mathbf{\$500} + \$500(1 + .08/12) + \$500(1 + .08/12)^{2} + \ldots + \$500(1 + .08/12)^{60} - \mathbf{\$500}$$

Except for the last term, we have a geometric series with a = $500, r = (1 + .08/12), and n = 60. Therefore the sum is

$$A = \frac{\$500[(1 + .08/12)^{61} - 1]}{.08/12} - \$500$$

$$A = \$500(74.9667) - \$500$$

$$A = \$37483.35 - \$500$$

$$A = \$36983.35$$

So, in the case of an annuity due, to find the future value, we increase the number of periods n by 1, and subtract one payment.

The Future Value of an **"Annuity Due"**
$$A = \frac{m[(1 + r/n)^{nt+1} - 1]}{r/n} - m$$

Most of the problems we are going to do in this chapter involve ordinary annuities, therefore, we will down play the significance of the last formula for the annuity due. We mentioned the formula for the annuity due only for completeness.

SECTION 6.3 SUMMARY

Finally, it is the author's wish that the student learn the concepts in a way that he or she will not have to memorize every formula. It is for this reason formulas are kept at a minimum. But before we conclude this section we will once again mention one single equation that will help us find the future value, as well as the sinking fund payment.

The Equation to Find the Future Value of an Ordinary Annuity, or the Amount of Periodic Payment to a Sinking Fund

If a payment of m dollars is made in an account n times a year at an interest r, then the future value A after t years is

$$A = \frac{m[(1 + r/n)^{nt} - 1]}{r/n}$$

The future value is also called the accumulated value.

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

SECTION 6.3 PROBLEM SET:ANNUITIES AND SINKING FUNDS

Each of the following problems involve an annuity - a sequence of payments.

1) Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly.	2) How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years?
3) At the end of each month Rita deposits $300 in an account that pays 5%. What will the final amount be in 4 years?	4) Mr. Chang wants to retire in 10 years and can save $650 every three months. If the interest rate is 7.8%, how much will he have (a) at the end of 5 years? (b) at the end of 10 years?
5) A firm needs to replace most of its machinery in five years at a cost of $500,000. The company wishes to create a sinking fund to have this money available in five years. How much should the quarterly deposits be if the fund earns 8%?	6) Mrs. Brown needs $5,000 in three years. If the interest rate is 9%, how much should she save at the end of each month to have that amount in three years?

SECTION 6.3 PROBLEM SET: ANNUITIES AND SINKING FUNDS

Each of the following problems involve an annuity - a sequence of payments.

7) A company has a $120,000 note due in 4 years. How much should be deposited at the end of each quarter in a sinking fund to payoff the note in four years if the interest rate is 8%?	8) You are now 20 years of age and decide to save $100 at the end of each month until you are 65. If the interest rate is 9.2%, how much money will you have when you are 65?
9) Is it better to receive $400 at the beginning of each month for six years, or a lump sum of $25,000 today if the interest rate is 7%? Explain.	10) To save money for a vacation, Jill decided to save $125 at the beginning of each month for the next 8 months. If the interest rate is 7%, how much money will she have at the end of 8 months?
11) Mrs. Gill puts $2200 at the end of each year in her IRA account that earns 9% per year. How much total money will she have in this account after 20 years?	12) If the inflation rate stays at 6% per year for the next five years, how much will the price be of a $15,000 car in five years? How much must you save at the end of each month at an interest rate of 7.3% to buy that car in 5 years?

6.4 Present Value of an Annuity and Installment Payment

In this section, you will learn to:

1. *Find the present value of an annuity.*
2. *Find the amount of installment payment on a loan.*

PRESENT VALUE OF AN ANNUITY

In Section 6.2, we learned to find the future value of a lump sum, and in Section 6.3, we learned to find the future value of an annuity. With these two concepts in hand, we will now learn to amortize a loan, and to find the present value of an annuity.

The **present value** of an annuity is the amount of money we would need now in order to be able to make the payments in the annuity in the future. In other word, the present value is the value now of a future stream of payments.

We start by breaking this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in Sections 6.2 and 6.3.

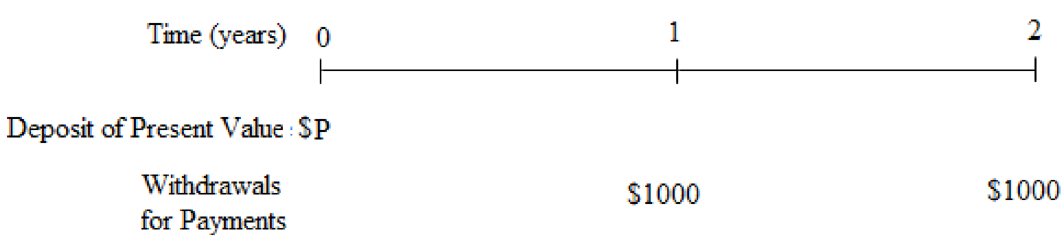
Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a $1000 bonus at the end of this year and another $1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings account.

How much money should Carlos put into the savings account now so that he will be able to withdraw $1000 one year from now and another $1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case we want to put a lump sum into the savings account now, so that lump sum is our principal, P. Then we want to withdraw that amount as a series of period payments; in this case the withdrawals are an annuity with $1000 payments at the end of each of two years.

We need to determine the amount we need in the account now, the present value, to be able to make withdraw the periodic payments later.



We use the compound interest formula from Section 6.2 with r = 0.04 and n = 1 for annual compounding to determine the present value of each payment of $1000.

Consider the first payment of $1000 at the end of year 1. Let P_1 be its present value

$$\$1000 = P_1(1.04)^1 \text{ so } P_1 = \$961.54$$

Now consider the second payment of $1000 at the end of year 2. Let P_2 is its present value

$$\$1000 = P_2(1.04)^2 \text{ so } P_2 = \$924.56$$

To make the $1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value $P = P_1 + P_2 = \$961.54 + \$924.56 = \$1886.10$

The calculation above was useful to illustrate the meaning of the present value of an annuity. But it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step by step calculation would be long and tedious.

Example 1 investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

◆***Example 1*** Suppose you have won a lottery that pays $1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

Solution: This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the problems to follow.

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of $1,000 per month for the next 20 years. Mr. Credit is happy with his $1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts P dollars, then the P dollars deposited at 8% for 20 years should yield the same amount as the $1,000 monthly payments for 20 years.
In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to equal.

Since Mr. Cash is receiving a lump sum of x dollars, its future value is given by the lump sum formula we studied in Section 6.2, and it is

$$A = P(1 + .08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of $1,000 per month, its future value is given by the annuity formula we learned in Section 6.3. This value is

$$A = \frac{\$1000\,[(1 + .08/12)^{240} - 1]}{.08/12}$$

The only way Mr. Cash will agree to the amount he receives is if these two future values are equal. So we set them equal and solve for the unknown.

$$P(1 + .08/12)^{240} = \frac{\$1000[(1 + .08/12)^{240} - 1]}{.08/12}$$

$$P(4.9268) = \$1000(589.02041)$$

$$P(4.9268) = \$589020.41$$

$$P = \$119,554.36$$

The present value of an ordinary annuity of $1,000 each month for 20 years at 8% is $119,554.36

The reader should also note that if Mr. Cash takes his lump sum of P = $119,554.36 and invests it at 8% compounded monthly, he will have an accumulated value of A=$589,020.41 in 20 years.

INSTALLMENT PAYMENT ON A LOAN

If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan.

They receive the loan amount called the principal (or present value) now and are obligated to pay back the principal in the future over a stated amount of time (term of the loan), as regular periodic payments with interest.

Example 2 examines how to calculate the loan payment, using reasoning similar to Example 1.

◆***Example 2*** Find the monthly payment for a car costing $15,000 if the loan is amortized over five years at an interest rate of 9%.

Solution: Again, consider the following scenario:

Two people, Mr. Cash and Mr. Credit, go to buy the same car that costs $15,000. Mr. Cash pays cash and drives away, but Mr. Credit wants to make monthly payments for five years.

Our job is to determine the amount of the monthly payment. We reason as follows:

If Mr. Credit pays m dollars per month, then the m dollar payment deposited each month at 9% for 5 years should yield the same amount as the $15,000 lump sum deposited for 5 years.

Again, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like them to be the same.

Since Mr. Cash is paying a lump sum of $15,000, its future value is given by the lump sum formula, and it is

$$\$15,000(1 + .09/12)^{60}$$

Mr. Credit wishes to make a sequence of payments, or an annuity, of x dollars per month, and its future value is given by the annuity formula, and this value is

$$\frac{x[(1 + .09/12)^{60} - 1]}{.09/12}$$

We set the two future amounts equal and solve for the unknown.

$$\$15{,}000(1 + .09/12)^{60} = \frac{m[(1+.09/12)^{60} - 1]}{.09/12}$$

$$\$15{,}000(1.5657) = m(75.4241)$$

$$\$311.38 = m$$

Therefore, the monthly payment needed to repay the loan is $311.38 for five years.

SECTION 6.4 SUMMARY

We summarize the method used in examples 1 and 2 below.

The Equation to Find the Present Value of an Annuity,

Or the Installment Payment for a Loan

If a payment of m dollars is made in an account n times a year at an interest r, then the present value P of the annuity after t years is

$$\mathbf{P(1 + r/n)^{nt} = \frac{m[(1 + r/n)^{nt} - 1]}{r/n}}$$

When used for a loan, the amount P is the loan amount, and m is the periodic payment needed to repay the loan over a term of t years with n payments per year.

If the present value or loan amount is needed, solve for P

If the periodic payment is needed, solve for m.

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

Finally, we note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula : $P(1 + r/n)^{nt} = \dfrac{m[(1 + r/n)^{nt} - 1]}{r/n}$ (Formula 6.4.1)

and solving for the present value P after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for P in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for P.

Alternate Method to find Present Value of an Annuity

Starting with formula 6.4.1: $P(1 + r/n)^{nt} = \dfrac{m[(1 + r/n)^{nt} - 1]}{r/n}$

Divide both sides by $(1+r/n)^{nt}$ to isolate P, and simplify

$$P = \frac{m[(1+r/n)^{nt} - 1]}{r/n} \cdot \frac{1}{(1+r/n)^{nt}}$$

$$P = \frac{m[1-(1+r/n)^{-nt}]}{r/n} \qquad \text{(Formula 6.4.2)}$$

The authors of this book believe that it is easier to use formula 6.4.1 at the top of this page and solve for P or m as needed. In this approach there are fewer formulas to understand, and many students find it easier to learn. In the problems the rest of this chapter, when a problem requires the calculation of the present value of an annuity, formula 6.4.1 will be used.

However, some people prefer formula 6.4.2, and it is mathematically correct to use that method. Note that if you choose to use formula 6.4.2, you need to be careful with the negative exponents in the formula. And if you needed to find the periodic payment, you would still need to do the algebra to solve for the value of m.

It would be a good idea to check with your instructor to see if he or she has a preference. In fact, you can usually tell your instructor's preference by noting how he or she explains and demonstrates these types of problems in class.

SECTION 6.4 PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

1) Shawn has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 8.2%, how much money can he hope to get?	2) Sonya bought a car for $15,000. Find the monthly payment if the loan is to be amortized over 5 years at a rate of 10.1%.
3) You determine that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for a car if the interest rate is 9% and you want to repay the loan in 5 years?	4) Compute the monthly payment for a house loan of $200,000 to be financed over 30 years at an interest rate of 10%.
5) If the $200,000 loan in the previous problem is financed over 15 years rather than 30 years at 10%, what will the monthly payment be?	6) Friendly Auto offers Jennifer a car for $2000 down and $300 per month for 5 years. Jason wants to buy the same car but wants to pay cash. How much must Jason pay if the interest rate is 9.4%?

SECTION 6.4 PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

7) The Gomez family bought a house for \$450,000. They paid 20% down and amortized the rest at 5.2% over a 30-year period. Find their monthly payment.	8) Mr. and Mrs. Wong purchased their new house for \$350,000. They made a down payment of 15%, and amortized the rest over 30 years. If the interest rate is 5.8%, find their monthly payment.
9) A firm needs a piece of machinery that has a useful life of 5 years. It has an option of leasing it for \$10,000 a year, or buying it for \$40,000 cash. If the interest rate is 10%, which choice is better?	10) Jackie wants to buy a \$19,000 car, but she can afford to pay only \$300 per month for 5 years. If the interest rate is 6%, how much does she need to put down?
11) Vijay's tuition at college for the next year is \$32,000. His parents have decided to pay the tuition by making nine monthly payments. If the interest rate is 6%, what is the monthly payment?	12) Glen borrowed \$10,000 for his college education at 8% compounded quarterly. Three years later, after graduating and finding a job, he decided to start paying off his loan. If the loan is amortized over five years at 9%, find his monthly payment for the next five years.

6.5 Miscellaneous Application Problems

In this section, you will learn to apply to concepts for compound interest for savings and annuities to:

1. *Find the outstanding balance, partway through the term of a loan, of the future payments still remaining on the loan.*
2. *Perform financial calculations in situations involving several stages of savings and/or annuities.*
3. *Find the fair market value of a bond.*
4. *Construct an amortization schedule for a loan.*

We have already developed the tools to solve most finance problems.
Now we use these tools to solve some application problems.

OUTSTANDING BALANCE ON A LOAN

One of the most common problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long term loans are paid off prematurely, we are often confronted with this problem.

To find the outstanding balance of a loan at a specified time, we need to find the present value P of all future payments that have not yet been paid. In this case t does not represent the entire term of the loan. Instead:

- t represents the time that still remains on the loan
- nt represents the total number of future payments.

◆***Example 1*** Mr. Jackson bought his house in 1995, and financed the loan for 30 years at an interest rate of 7.8%. His monthly payment was $1260. In 2015, Mr. Jackson decides to pay off the loan. Find the balance of the loan he still owes.

Solution: The reader should note that the original amount of the loan is not mentioned in the problem. That is because we don't need to know that to find the balance.

The original loan was for 30 years. 20 years have past so there are
years still remaining. 12(10) = 120 payments still remain to be paid on this loan.

As for the bank or lender is concerned, Mr. Jackson is obligated to pay $1260 each month for 10 more years; he still owes a total of 120 payments. But since Mr. Jackson wants to pay it all off now, we need to find the present value P at the time of repayment of the remaining 10 years of payments of $1260 each month.
Using the formula we get for the present value of an annuity, we get

$$P(1 + .078/12)^{120} = \frac{\$1260\left[\left(1+.078/12\right)^{120}-1\right)]}{(.078/12)}$$

$$P\,(2.17597) = \$227957.85$$

$$P = \$104761.48$$

To Find the Outstanding Balance of a Loan

If a loan has a payment of m dollars made i n times a year at an interest r,
then the outstanding value of the loan when there are t years still remaining on the loan is given by P:

$$P(1 + r/n)^{nt} = \frac{m[(1 + r/n)^{nt} – 1]}{r/n}$$

IMPORTANT: Note that t is not the original term of the loan but instead
t is the amount of time still remaining in the future
nt is the number of payments still remaining in the future

If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated BEFORE using the above formula as
t = original term of loan – time already passed since the start date of the loan.

Note that there are other methods to find the outstanding balance on a loan, but the method illustrated above is the easiest.

One alternate method would be to use an amortization schedule, as illustrated toward the end of this section. An amortization schedule shows the payments, interest, and outstanding balance step by step after each loan payment. An amortization schedule is tedious to calculate by hand but can be easily constructed using spreadsheet software.

Another way to find the outstanding balance, that we will not illustrate here, is to find the difference A – B, where

A = the original loan amount (principal) accumulated to the date on which we want to find the outstanding balance (using compound interest formula)

B = the accumulated value of all payments that have been made as of the date on which we want to find the outstanding balance (using formula for accumulated value of an annuity)

In this case we would need do a compound interest calculation and an annuity calculation; we then need to find the difference between them. Three calculations are needed instead of one. It is a mathematically acceptable way to calculate the outstanding balance. However,
it is very strongly recommended that students use the method explained in box above and illustrated in Example 1, as it is much simpler.

PROBLEMS INVOLVING MULTIPLE STAGES OF SAVINGS AND/OR ANNUITIES

Consider the following situations:

a. Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

b. Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period. ; the accumulated value from first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

◆***Example 2*** Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college; then it is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much can Aisha withdraw from the account every six months while she is in college?

Solution: **Part 1: Accumulation of College Savings:** Find the accumulated value at the end of 18 years of a sum of $8000 invested at 5% compounded semiannually.

$$A = \$8000(1 + .05/2)^{(2\times 18)} = \$8000(1.025)^{36} = \$8000(2.432535)$$

$$A = \$19460.28$$

Part 2: Seminannual annuity payout from savings to put toward college expenses. Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.

A= $19460.28 in Part 1 is the accumulated value at the end of the savings period. This ecomes the present value P =$19460.28 when calculating the semiannual payments in Part 2.

$$\$19460.28\left(1+\frac{.05}{2}\right)^{2\times 4} = \frac{m\left[\left(1+\frac{.05}{2}\right)^{2\times 4}-1\right]}{(.05/2)}$$

$$\$23710.46 = m\,(8.73612)$$

$$m = \$2714.07$$

Aisha will be able to withdraw $2714.07 semiannually for her college expenses.

◆***Example 3*** Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

Solution: **Part 1: Accumulation of Retirement Savings:** Find the accumulated value at the end of 30 years of $1000 deposited at the end of each quarter into a retirement savings account earning 6% interest compounded quarterly.

$$A = \frac{\$1000\left[\left(1+.06/4\right)^{4\times 30}-1\right]}{(.06/4)}$$

A = $331288.19

Part 2: Monthly retirement annuity payout: Find the amount of the monty annuity payments for 25 years using the accumulated savings from part 1 of the problem with an interest rate of 4.8% compounded monthly.

A= $331288.19 in Part 1 is the accumulated value at the end of the savings period. This amount will become the present value P =$331288.19 when calculating the monthly retirement annuity payments in Part 2.

$$\$331288.19\left(1+.048/12\right)^{12\times 25} = \frac{m\left[\left(1+.048/12\right)^{12\times 25}-1\right]}{(.048/12)}$$

$1097285.90 = m (578.04483)

m = $1898.27

Aisha will have a monthly retirement annuity income of $1898.27 when she retires.

FAIR MARKET VALUE OF A BOND

Whenever a business, and for that matter the U. S. government, needs to raise money it does it by selling bonds. A **bond** is a certificate of promise that states the terms of the agreement. Usually the business sells bonds for the **face amount** of $1,000 each for a stated **term**, a period of time ending at a specified **maturity** date.

The person who buys the bond, the **bondholder**, pays $1,000 to buy the bond.

The bondholder is promised two things: First that he will get his $1,000 back at the maturity date, and second that he will receive a fixed amount of interest every six months.

As the market interest rates change, the price of the bond starts to fluctuate. The bonds are bought and sold in the market at their **fair market value**.

The interest rate a bond pays is fixed, but if the market interest rate goes up, the value of the bond drops since the money invested in the bond could earn more if invested elsewhere. When the value of the bond drops, we say it is trading at a **discount**.

On the other hand, if the market interest rate drops, the value of the bond goes up since the bond now yields a higher return than the market interest rate, and we say it is trading at a **premium**.

◆***Example 4*** The Orange Computer Company needs to raise money to expand. It issues a 10-year $1,000 bond that pays $30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?

Solution: The bond certificate promises us two things – An amount of $1,000 to be paid in 10 years, and a semi-annual payment of $30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of $1,000 we are to receive in 10 years, as well as, the present value of the $30 semi-annual payments for the 10 years.

We will let P_1 = the present value of the (face amount of $1,000

$$P_1(1 + .07/2)^{20} = \$1{,}000$$

Since the interest is paid twice a year, the interest is compounded twice a year and nt = 2(10)=20

$$P_1(1.9898) = \$1{,}000$$

$$P_1 = \$502.56$$

We will let P_2 = the present value of the $30 semi-annual payments is

$$P_2(1 + .07/2)^{20} = \frac{\$30[(1+.07/2)^{20}-1]}{(.07/2)}$$

$$P_2\ (1.9898) = 848.39$$

$$P_2 = \$426.37$$

The present value of the lump-sum $1,000 = $502.56

The present value of the $30 semi-annual payments = $426.37

The fair market value of the bond is $P = P_1 + P_2$ = $502.56 + $426.37 = $928.93

Note that because the market interest rate of 7% is higher than the bond's implied interest rate of 6% implied by the semiannual payments, the bond is selling at a discount; its fair market value of $928.93 is less than its face value of $1000.

◆***Example 5*** A state issues a 15 year $1000 bond that pays $25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?

Solution: The bond certificate promises two things – an amount of $1,000 to be paid in 15 years, and semi-annual payments of $25 for 15 years. To find the fair market value of the bond, we find the present value of the $1,000 face value we are to receive in 15 years and add it to the present value of the $25 semi-annual payments for the 15 years. In this example, nt = 2(15)=30.

We will let P_1 = the present value of the lump-sum $1,000

$$P_1(1 + .04/2)^{30} = \$1,000$$

$$P_1 = \$552.07$$

We will let P_2 = the present value of the $25 semi-annual payments is

$$P_2\,(1 + .04/2)^{30} = \frac{\$25[(1+.04/2)^{30} - 1]}{(.04/2)}$$

$$P_2\ (1.18114) = \$1014.20$$

$$P_2 = \$559.90$$

The present value of the lump-sum $1,000 = $552.07

The present value of the $30 semi-annual payments = $559.90

Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of $1,111.97 is more than the face value of $1,000.

To summarize:

To Find the Fair Market Value of a Bond:

Find the present value of the face amount A that is payable at the maturity date:

$$\mathbf{A = P_1\,(1 + r/n)^{nt}} \quad \text{; solve to find } P_1$$

Find the present value of the semiannually payments of $m over the term of the bond:

$$\mathbf{P_2(1 + r/n)^{nt} = \frac{m[(1 + r/n)^{nt} - 1]}{r/n}} \quad \text{; solve to find } P_2$$

The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$$P = \mathbf{P_1 + P_2}$$

AMORTIZATION SCHEDULE FOR A LOAN

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

◆***Example 6*** An amount of $500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Solution: The reader can verify that the monthly payment is $86.27.

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is

(outstanding balance)(the monthly interest rate) = ($500)(.12/12) = $5

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 – $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is ($418.73)(.12/12) = $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 – $82.08 = $336.65. The process continues in the table below.

Payment #	Payment	Interest	Debt Payment	Balance
1	$86.27	$5	$81.27	$418.73
2	$86.27	$4.19	$82.08	$336.65
3	$86.27	$3.37	$82.90	$253.75
4	$86.27	$2.54	$83.73	$170.02
5	$86.27	$1.70	$84.57	$85.45
6	$86.27	$0.85	$85.42	$0.03

Note that the last balance of 3 cents is due to error in rounding off.

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30 year mortgage loan with monthly payments would have (12)(30)=360 rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have 12(5)=60 rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

Most of the other applications in this section's problem set are reasonably straightforward, and can be solved by taking a little extra care in interpreting them. And remember, there is often more than one way to solve a problem.

SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

For problems 1 - 4, assume a $200,000 house loan is amortized over 30 years at an interest rate of 5.4%.

1) Find the monthly payment.	2) Find the balance owed after 20 years.
3) Find the balance of the loan after 100 payments.	4) Find the monthly payment if the original loan were amortized over 15 years.

5) Mr. Patel wants to pay off his car loan. The monthly payment for his car is $365, and he has 16 payments left. If the loan was financed at 6.5%, how much does he owe?	6) An amount of $2000 is borrowed for a year at a rate of 7%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment going toward reducing the debt, and the balance.

SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

7) Fourteen months after Dan bought his new car he lost his job. His car was repossessed by his lender after he made only 14 monthly payments of $376 each. If the loan was financed over a 4-year period at an interest rate of 6.3%, how much did the car cost the lender? In other words, how much did Dan still owe on the car?

8) You have a choice of either receiving $5,000 at the end of each year for the next 5 years or receiving $3000 per year for the next 10 years. If the current interest rate is 9%, which is better?

9) Mr. Smith is planning to retire in 25 years and would like to have $250,000 then. What monthly payment made at the end of each month to an account that pays 6.5% will achieve his objective?

10) Assume Mr. Smith has reached retirement and has $250,000 in an account which is earning 6.5%. He would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment.

11) Mrs. Garcia is planning to retire in 20 years. She starts to save for retirement by depositing $2000 each quarter into a retirement investment account that earns 6% interest compounded quarterly. Find the accumulated value of her retirement savings at the end of 20 years.

12) Assume Mrs. Garcia has reached retirement and has accumulated the amount found in question 13 in a retirement savings account. She would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. Assume the account now pays 5.4% compounded monthly.

SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

13) A ten-year $1,000 bond pays $35 every six months. If the current interest rate is 8.2%, find the fair market value of the bond.

Hint: You must do the following.

a) Find the present value of $1000.

b) Find the present value of the $35 payments.

c) The fair market value of the bond = a + b

14) Find the fair market value of the ten-year $1,000 bond that pays $35 every six months, if the current interest rate has dropped to 6%.

Hint: You must do the following.

a) Find the present value of $1000.

b) Find the present value of the $35 payments.

c) The fair market value of the bond = a + b

15) A twenty-year $1,000 bond pays $30 every six months. If the current interest rate is 4.2%, find the fair market value of the bond.

Hint: You must do the following.

a) Find the present value of $1000.

b) Find the present value of the $30 payments.

c) The fair market value of the bond = a + b

16) Find the fair market value of the twenty-year $1,000 bond that pays $30 every six months, if the current interest rate has increased to 7.5%.

SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

17) Mr. and Mrs. Nguyen deposit $10,000 into a college investment account when their new baby grandchild is born. The account earns 6.25% interest compounded quarterly.

a, When their grandchild reaches the age of 18, what is the accumulated value of the college investment account?

b, The Nguyen's grandchild has just reached the age of 18 and started college. If she is to withdraw the money in the college savings account n equal monthly payments over the next 4 years, how much money will be withdrawn each month?

18) Mr. Singh is 38 and plans to retire at age 65. He opens a retirement savings account.

a. Mr. Singh wants to save enough money to accumulate $500,000 by the time he retires. The retirement investment account pays 7% interest compounded monthly. How much does he need to deposit each month to achieve this goal?

b. Mr. Singh has now reached at 65 and retires. How much money can he withdraw each month for 25 years if the retirement investment account now pays 5.2% interest, compounded monthly?

6.6 Classification of Finance Problems

In this section, you will review the concepts of chapter 6 to:

1. *re-examine the types of financial problems and classify them.*
2. *re-examine the vocabulary words used in describing financial calculations*

We'd like to remind the reader that the hardest part of solving a finance problem is determining the category it falls into. So in this section, we will emphasize the classification of problems rather than finding the actual solution.

We suggest that the student read each problem carefully and look for the word or words that may give clues to the kind of problem that is presented. For instance, students often fail to distinguish a lump-sum problem from an annuity. Since the payments are made each period, an annuity problem contains words such as each, every, per etc.. One should also be aware that in the case of a lump-sum, only a single deposit is made, while in an annuity numerous deposits are made at equal spaced time intervals. To help interpret the vocabulary used in the problems, we include a glossary at the end of this section.

Students often confuse the present value with the future value. For example, if a car costs \$15,000, then this is its present value. Surely, you cannot convince the dealer to accept \$15,000 in some future time, say, in five years. Recall how we found the installment payment for that car. We assumed that two people, Mr. Cash and Mr. Credit, were buying two identical cars both costing \$15, 000 each. To settle the argument that both people should pay exactly the same amount, we put Mr. Cash's cash of \$15,000 in the bank as a lump-sum and Mr. Credit's monthly payments of x dollars each as an annuity. Then we make sure that the future values of these two accounts are equal. As you remember, at an interest rate of 9%

the future value of Mr. Cash's lump-sum was $\$15{,}000(1 + .09/12)^{60}$, and

the future value of Mr. Credit's annuity was $\dfrac{x[(1+.09/12)^{60}-1]}{.09/12}$.

To solve the problem, we set the two expressions equal and solve for m.

The present value of an annuity is found in exactly the same way. For example, suppose Mr. Credit is told that he can buy a particular car for \$311.38 a month for five years, and Mr. Cash wants to know how much he needs to pay. We are finding the present value of the annuity of \$311.38 per month, which is the same as finding the price of the car. This time our unknown quantity is the price of the car. Now suppose the price of the car is P, then

the future value of Mr. Cash's lump-sum is $P(1 + .09/12)^{60}$, and

the future value of Mr. Credit's annuity is $\dfrac{\$311.38[(1 + .09/12)^{60} - 1]}{.09/12}$.

Setting them equal we get,

$$P(1 + .09/12)^{60} = \frac{\$311.38[(1 + .09/12)^{60} - 1]}{.09/12}$$

$$P(1.5657) = (\$311.38)(75.4241)$$

$$P(1.5657) = \$23{,}485.57$$

$$P = \$15{,}000.04$$

CLASSIFICATION OF PROBLEMS AND EQUATIONS FOR SOLUTIONS

We now list six problems that form a basis for all finance problems. Further, we classify these problems and give an equation for the solution.

◆***Problem 1*** If $2,000 is invested at 7% compounded quarterly, what will the final amount be in 5 years?

Classification: **Future (accumulated) Value of a Lump-sum** or FV of a lump-sum.

Equation: $FV = A = \$2000(1 + .07/4)^{20}$.

◆***Problem 2*** How much should be invested at 8% compounded yearly, for the final amount to be $5,000 in five years?

Classification: **Present Value of a Lump-sum** or PV of a lump-sum.

Equation: $PV(1 + .08)^5 = \$5{,}000$

◆***Problem 3*** If $200 is invested *each* month at 8.5% compounded monthly, what will the final amount be in 4 years?

Classification: **Future (accumulated) Value of an Annuity** or FV of an annuity.

Equation: $FV = A = \dfrac{\$200[(1 + .085/12)^{48} - 1]}{.085/12}$

◆***Problem 4*** How much should be invested *each* month at 9% for it to accumulate to $8,000 in three years?

Classification: **Sinking Fund Payment**

Equation: $\dfrac{m[(1 + .09/12)^{36} - 1]}{.09/12} = \$8{,}000$

◆***Problem 5*** Keith has won a lottery paying him $2,000 *per* month for the next 10 years. He'd rather have the entire sum now. If the interest rate is 7.6%, how much should he receive?

Classification: **Present Value of an Annuity** or PV of an annuity.

Equation: $PV(1 + .076/12)^{120} = \dfrac{\$2000[(1 + .076/12)^{120} - 1]}{.076/12}$

◆***Problem 6*** Mr. A has just donated $25,000 to his alma mater. Mr. B would like to donate an equivalent amount, but would like to pay by monthly payments over a five year period. If the interest rate is 8.2%, determine the size of the monthly payment?

Classification: **Installment Payment**.

Equation: $\dfrac{m[(1 + .082/12)^{60} - 1]}{.082/12} = \$25{,}000(1 + .082/12)^{60}$.

GLOSSARY: VOCABULARY AND SYMBOLS USED IN FINANCIALCALCULATIONS

As we've seen in these examples, it's important to read the problems carefully to correctly identify the situation. It is essential to understand to vocabulary for financial problems. Many of the vocabulary words used are listed in the glossary below for easy reference.

t	Term	Time period for a loan or investment. In this book t is represented in years and should be converted into years when it is stated in months or other units.
P	Principal	Principal is the amount of money borrowed in a loan. If a sum of money is invested for a period of time, the sum invested at the start is the Principal.
P	Present Value	Value of money at the beginning of the time period.
A	Accumulated Value Future Value	Value of money at the end of the time period
D	Discount	In loans involving simple interest, a discount occurs if the interest is deducted from the loan amount at the beginning of the loan period, rather than being repaid at the end of the loan period.
m	Periodic Payment	The amount of a constant periodic payment that occurs at regular intervals during the time period under consideration (examples: periodic payments made to repay a loan, regular periodic payments into a bank account as savings, regular periodic payment to a retired person as an annuity,)
n	Number of payment periods and compounding periods per year	In this book, when we consider periodic payments, we will always have the compounding period be the same as the payment period. In general the compounding and payment periods do not have to be the same, but the calculations are more complicated if they are different. If the periods differ, formulas for the calculations can be found in finance textbooks or various online resources. Calculations can easily be done using technology such as an online financial calculator, or financial functions in a spreadsheet, or a financial pocket calculator.
nt	Number of periods	nt = (number of periods per year)×(number of years) nt gives the total number of payment and compounding periods In some situations we will calculate nt as the multiplication shown above. In other situations the problem may state nt, such as a problem describing an investment of 18 months duration compounded monthly. In this example: nt = 18 months and n = 12; then t = 1.5 years but t is not stated explicitly in the problem. The TI-84+ calculators built in TVM solver uses N = nt.
r	Annual interest rate Nominal rate	The stated annual interest rate. This is stated as a percent but converted to decimal form when using financial calculation formulas. If a bank account pays 3% interest compounded quarterly, then 3% is the nominal rate, and it is included in the financial formulas as r = 0.03
r/n	Interest rate per compounding period	If a bank account pays 3% interest compounded quarterly, then r/n = 0.03/4 = 0. 075, corresponding to a rate of 0.75% per quarter. Some Finite Math books use the symbol i to represent r/n

r_{EFF}	Effective Rate Effective Annual Interest Rate APY Annual Percentage Yield APR Annual Percentage Rate	The effective rate is the interest rate compounded annually that would give the same interest rate as the compounded rate stated for the investment. The effective rate provides a uniform way for investors or borrowers to compare different interest rates with different compounding periods.
I	Interest	Money paid by a borrower for the use of money borrowed as a loan. Money earned over time when depositing money into a savings account, certificate of deposit, or money market account. When a person deposits money in a bank account, the person depositing the funds is essentially temporarily lending the money to the bank and the bank pays interest to the depositor.
	Sinking Fund	A fund set up by making payments over a period of time into a savings or investment account in order to save to fund a future purchase. Businesses use sinking funds to save for a future purchase of equipment at the end of the savings period by making periodic installment payments into a sinking fund.
	Annuity	An annuity is a stream of periodic payments. In this book it refers to a stream of constant periodic payments made at the end of each compounding period for a specific amount of time. In common use the term annuity generally refers to a constant stream of periodic payments received by a person as retirement income, such as from a pension. Annuity payments in general may be made at the end of each payment period (ordinary annuity) or at the start of each period (annuity due). The compounding periods and payment periods do not need to be equal, but in this textbook we only consider situations when these periods are equal.
	Lump Sum	A single sum of money paid or deposited at one time, rather than being spread out over time. An example is lottery winnings if the recipient chooses to receive a single "lump sum" one-time payment, instead of periodic payments over a period of time or as. Use of the word lump sum indicates that this is a one time transaction and is not a stream of periodic payments.
	Loan	An amount of money that is borrowed with the understanding that the borrower needs to repay the loan to the lender in the future by the end of a period of time that is called the term of the loan. The repayment is most often accomplished through periodic payments until the loan has been completely repaid over the term of the loan. However there are also loans that can be repaid as a single sum at the end of the term of the loan, with interest paid either periodically over the term or in a lump sum at the end of the loan or as a discount at the start of the loan.

SECTION 6.6 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

Let the letters A, B, C, D, E and F be represented as follows:

A = FV of a lump-sum	C = FV of an annuity	E = Installment payment
B = PV of a lump-sum	D = Sinking fund payment	F = PV of an annuity

Classify each by writing the appropriate letter in the box, and write an equation for solution.

1) What monthly deposits made to an account paying 9% will grow to $10,000 in 4 years?

2) An amount of $4000 is invested at 6% compounded daily. What will the final amount be in 5 years?

3) David has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum now. If the current interest rate is 7%, how much money can he hope to get?

4) Each month Linda deposits $250 in an account that pays 9%. How much money will she have in 4 years?

5) Find the monthly payment for a $15,000 car if the loan is amortized over 4 years at a rate of 10%.

6) What lump-sum deposited in an account paying 7% compounded daily will grow to $10,000 in 5 years?

7) What amount of quarterly payments will amount to $250,000 in 5 years at a rate of 8%?

8) The Chang family bought their house 25 years ago. They had their loan financed for 30 years at an interest rate of 11% resulting in a payment of $1350 a month. Find the balance of the loan.

9-10) A 10-year $1000 bond pays $35 every six months. If the current interest rate is 8%, in order to find the fair market value of the bond, we need to find the following.

9) The present value of $1000.

10) The present value of the $35 per six month payments.

SECTION 6.6 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

A = FV of a lump-sum	C = FV of an annuity	E = Installment payment
B = PV of a lump-sum	D = Sinking fund payment	F = PV of an annuity

11) What lump-sum deposit made today is equal to 33 monthly deposits of $500 if the interest rate is 8%?

12) What monthly deposits made to an account paying 10% will accumulated to $10,000 in six years?

13) A department store charges a finance charge of 1.5% per month on the outstanding balance. If Ned charged $400 three months ago and has not paid his bill, how much does he owe?

14) What will the value of $300 monthly deposits be in 10 years if the account pays 12% compounded monthly?

15) What lump-sum deposited at 6% compounded daily will grow to $2000 in three years?

16) A company buys an apartment complex for $5,000,000 and amortizes the loan over 10 years. What is the yearly payment if the interest rate is 14%?

17) In 2002, a house in Rock City cost $300,000. Real estate in Rock City has been increasing in value at the annual rate of 5.3%.. Find the price of that house in 2016.

18) You determine that you can afford to pay $400 per month for a car. What is the maximum price you can pay for a car if the interest rate is 11% and you want to repay the loan in 4 years?

19) A business needs $350,000 in 5 years. How much lump-sum should be put aside in an account that pays 9% so that five years from now the company will have $350,000?

20) A person wishes to have $500,000 in a pension fund 20 years from now. How much should he deposit each month in an account paying 9% compounded monthly?

SECTION 6.7 PROBLEM SET: CHAPTER REVIEW

1) Manuel borrows $800 for 6 months at 18% simple interest. How much does he owe at the end of 6 months?

2) The population of a city is 65,000 and expects to grow at a rate of 2.3% per year for the next 10 years. What will the population of this city be in 10 years?

3) The Gill family is buying a $250,000 house with a 10% down payment. If the loan is financed over a 30 year period at an interest rate of 4.8%, what is the monthly payment?

4) Find the monthly payment for the house in the above problem if the loan was amortized over 15 years.

5) You look at your budget and decide that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for the car if the interest rate is 8.6% and you want to finance the loan over 5 years?

6) Mr. Nakahama bought his house in the year 1998. He had his loan financed for 30 years at an interest rate of 6.2% resulting in a monthly payment of $1500. In 2015, 17 years later, he paid off the balance of the loan. How much did he pay?

7) Lisa buys a car for $16,500, and receives $2400 for her old car as a trade-in value. Find the monthly payment for the balance if the loan is amortized over 5 years at 8.5%.

8) A car is sold for $3000 cash down and $400 per month for the next 4 years. Find the cash value of the car today if the money is worth 8.3% compounded monthly.

9) An amount of $2300 is borrowed for 7 months at a simple interest rate of 16%. Find the discount and the proceeds.

10) Marcus has won a lottery paying him $5000 per month for the next 25 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 7.3%, how much money can he hope to get?

11) In the year 2000, an average house in Star City cost $250,000. If the average annual inflation rate for the past years has been about 4.7%, what was the price of that house in 2015?

12) Find the 'fair market' value of a ten-year $1000 bond which pays $30 every six months if the current interest rate is 7%. What if the current interest rate is 5%?

13) A Visa credit card company has a finance charge of 1.5% per month (18% per year) on the outstanding balance. John owed $3200 and has been delinquent for 5 months. How much total does he owe, now?

14) You want to purchase a home for $200,000 with a 30-year mortgage at 9.24% interest. Find a) the monthly payment and b) the balance owed after 20 years.

15) When Jose bought his car, he amortized his loan over 6 years at a rate of 9.2%, and his monthly payment came out to be $350 per month. He has been making these payments for the past 40 months and now wants to pay off the remaining balance. How much does he owe?

16) A lottery pays $10,000 per month for the next 20 years. If the interest rate is 7.8%, find both its present and future values.

SECTION 6.7 PROBLEM SET: CHAPTER REVIEW

17) A corporation estimates it will need $300,000 in 8 years to replace its existing machinery. How much should it deposit each quarter in a sinking fund earning 8.4% compounded quarterly to meet this obligation?

18) Our national debt in 1992 was about $4 trillion. If the annual interest rate was 7% then, what was the daily interest on the national debt?

19) A business must raise $400,000 in 10 years. What should be the size of the owners' monthly payments to a sinking fund paying 6.5% compounded monthly?

20) The population of a city of 80,000 is growing at a rate of 3.2% per year. What will the population be at the end of 10 years?

21) A sum of $5000 is deposited in a bank today. What will the final amount be in 20 months if the bank pays 9% and the interest is compounded monthly?

22) A manufacturing company buys a machine for $500 cash and $50 per month for the next 3 years. Find the cash value of the machine today if the money is worth 6.2% compounded monthly.

23) The United States paid about 4 cents an acre for the Louisiana Purchase in 1803. Suppose the value of this property grew at a rate of 5.5% annually. What would an acre be worth in the year 2000?

24) What amount should be invested per month at 9.1% compounded monthly so that it will become $5000 in 17 months?

25) A machine costs $8000 and has a life of 5 years. It can be leased for $160 per month for 5 years with a cash down payment of $750. The current interest rate is 8.3%. Is it cheaper to lease or to buy?

26) If inflation holds at 5.2% per year for 5 years, what will be the cost in 5 years of a car that costs $16,000 today? How much will you need to deposit each quarter in a sinking fund earning 8.7% per year to purchase the new car in 5 years?

27) City Bank pays an interest rate of 6%, while Western Bank pays 5.8% compounded continuously. Which one is a better deal?

28) Ali has inherited $20,000 and is planning to invest this amount at 7.9% interest. At the same time he wishes to make equal monthly withdrawals to use up the entire sum in 5 years. How much can he withdraw each month?

29) Jason has a choice of receiving $300 per month for the next 5 years or $500 per month for the next 3 years. Which one is worth more if the current interest rate is 7.7%?

30) If a bank pays 6.8% compounded continuously, how long will it take to double your money?

31) A mutual fund claims a growth rate of 8.3% per year. If $500 per month is invested, what will the final amount be in 15 years?

32) Mr. Vasquez has been given two choices for his compensation. He can have $20,000 cash plus $500 per month for 10 years, or he can receive $12,000 cash plus $1000 per month for 5 years. If the interest rate is 8%, which is the better offer?

SECTION 6.7 PROBLEM SET: CHAPTER REVIEW

33) How much should Mr. Shackley deposit in a trust account so that his daughter can withdraw $400 per month for 4 years if the interest rate is 8%?

34) Mr. Albers borrowed $425,000 from the bank for his new house at an interest rate of 4.7%. He will make equal monthly payments for the next 30 years. How much money will he end up paying the bank over the life of the loan, and how much is the interest?

35) Mr. Tong puts away $500 per month for 10 years in an account that earns 9.3%. After 10 years, he decides to withdraw $1,000 per month. If the interest rate stays the same, how long will it take Mr. Tong to deplete the account?

36) An amount of $5000 is borrowed for 15 months at an interest rate of 9%. Find the monthly payment and construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the amount of payment contributing towards debt, and the outstanding debt.

CHAPTER 7

Chapter 7: Sets and Counting

In this chapter, you will learn to:

1. *Use set theory and Venn diagrams to solve counting problems.*
2. *Use the Multiplication Axiom to solve counting problems.*
3. *Use Permutations to solve counting problems.*
4. *Use Combinations to solve counting problems.*
5. *Use the Binomial Theorem to expand $(x + y)^n$.*

7.1 Sets

In this section, you will learn to:

1. *Use set notation to represent unions, intersections, and complements of sets*
2. *Use Venn diagrams to solve counting problems.*

INTRODUCTION TO SETS

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

A **set** is a collection of objects, and its members are called the **elements** of the set. We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

C = {Ken, Bob, Tran, Shanti, Eric}

A set that has no members is called an **empty set** The empty set is denoted by the symbol ∅.

Set Equality ; Subsets

Two sets are **equal** if they have the same elements.

A set A is a **subset** of a set B if every member of A is also a member of B.

Suppose C = {Al, Bob, Chris, David, Ed} and A = {Bob, David}.
Then A is a subset of C, written as $A \subseteq C$.

Every set is a subset of itself, and the empty set is a subset of every set.

Union of Two Sets

Let A and B be two sets, then the union of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

Intersection of Two Sets

Let A and B be two sets, then the intersection of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

A **universal set** U is the set consisting of all elements under consideration.

Complement of a Set; Disjoint Sets

Let A be any set, then the complement of set A, written as $\overline{A}$, is the set consisting of elements in the universal set U that are not in A.

Two sets A and B are called disjoint sets if their intersection is an empty set. Clearly, a set and its complement are disjoint; however two sets can be disjoint and not be complements.

◆***Example 1*** List all the subsets of the set of primary colors { red, yellow, blue}.

Solution: The subsets are $\varnothing$, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

◆***Example 2*** Let F = { Aikman, Jackson, Rice, Sanders, Young },
and let B = {Griffey, Jackson, Sanders, Thomas}.
Find the intersection of the sets F and B.

Solution: The intersection of the two sets is the set whose elements belong to both sets.
Therefore, $F \cap B$ = {Jackson, Sanders }

◆***Example 3*** Find the union of the sets F and B given as follows.

F = { Aikman, Jackson, Rice, Sanders, Young }
B = {Griffey, Jackson, Sanders, Thomas}

Solution: The union of two sets is the set whose elements are either in A or in B or in both A and B. Observe that when writing the union of two sets, the repetitions are avoided.

$F \cup B$ = {Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young}

◆***Example 4*** Let the universal set U = {red, orange, yellow, green, blue, indigo, violet}, and P = {red, yellow, blue}. Find the complement of P.

Solution: The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$\overline{P}$ = {orange, green, indigo, violet}

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\overline{P}$ represents those colors of the spectrum that are not primary colors.

◆***Example 5*** Let the universal set U = {red, orange, yellow, green, blue, indigo, violet}, and P = {red, yellow, blue} . Find a set R so that R is not the complement of P but R and P are disjoint.

Solution: R = {orange, green } and P = (red, yellow, blue} are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However they are not complements because their union $P \cup R$ = {red, yellow, blue, orange, green} is not equal to the universal set U.

◆***Example 6*** Let U = {red, orange, yellow, green, blue, indigo, violet}, P = {red, yellow, blue}, Q = { red, green}, and R = {orange, green, indigo}. Find $\overline{P \cup Q} \cap \overline{R}$.

Solution: We do the problems in steps:

$P \cup Q$ = {red, yellow, blue, green}
$\overline{P \cup Q}$ = {orange, indigo, violet}
$\overline{R}$ = {red, yellow, blue, violet}
$\overline{P \cup Q} \cap \overline{R}$ = {violet}

VENN DIAGRAMS

We now use Venn diagrams to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

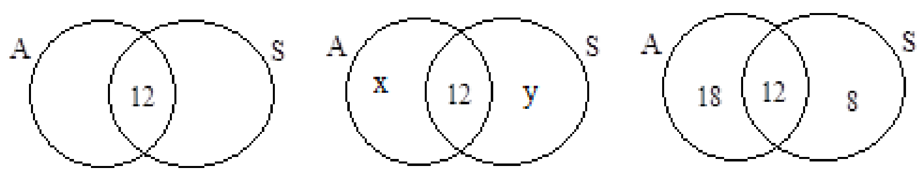
A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

◆***Example 7*** Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?

Solution: We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets.



Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that

$x + 12 = 30$, or $x = 18$.

Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements.

Thus, $y + 12 = 20$ which in turn makes $y = 8$.

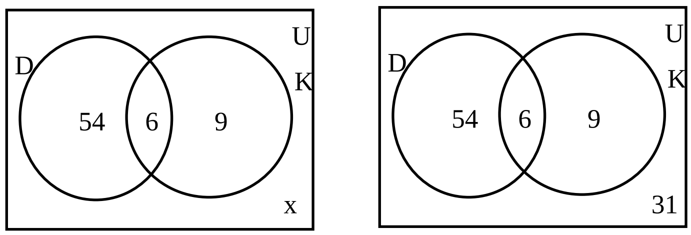
Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

Therefore, $18 + 12 + 8 = 38$ people took part in the survey.

◆***Example 8*** A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?

Solution: The problem is similar to the one in the previous example.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.



We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let x represent those people in the universal set that are neither in the set D nor in K. This means $54 + 6 + 9 + x = 100$, or $x = 31$.

Therefore, there are 31 people in the survey who have visited neither place.

◆***Example 9*** A survey of 100 exercise conscious people resulted in the following information:

- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

a. How many jog but do not swim or cycle?

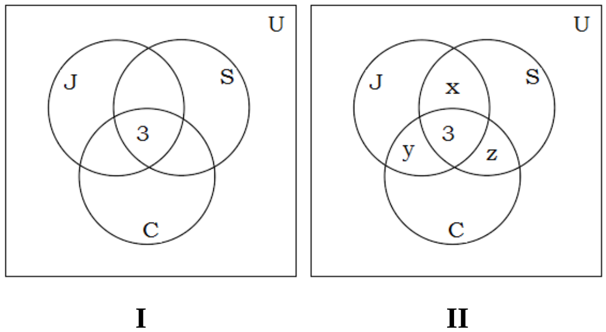
b. How many take part in only one of the activities?

c. How many do not take part in any of these activities?

Solution: Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



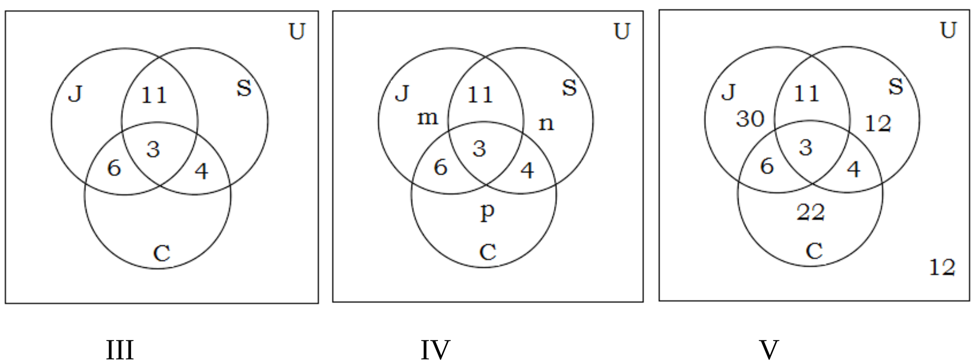
We place a 3 in the innermost region of figure I because it represents the number of people who participate in all three activities. Next we use figure II to compute x, y and z.

Since 14 people jog and swim, x + 3 = 14, or x = 11.

The fact that 9 people jog and cycle results in y + 3 = 9, or y = 6.

Since 7 people swim and cycle, z + 3 = 7, or z = 4.

This information is depicted in figure III.



Now we proceed to find the unknowns m, n and p, as shown in figure IV

Since 50 people jog, m + 11 + 6 + 3 = 50, or m = 30.

30 people swim, therefore, n + 11 + 4 + 3 = 30, or n = 12.

35 people cycle, therefore, p + 6 + 4 + 3 = 35, or p = 22.

By adding all the entries in all three sets, we get a sum of 88.
Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 – 88, or 12.

In figure V, all the information is sorted out, and the questions can readily be answered.

a. The number of people who jog but do not swim or cycle is 30.

b. The number who take part in only one of these activities is 30 + 12 + 22 = 64.

c. The number of people who do not take part in any of these activities is 12.

SECTION 7.1 PROBLEM SET: SETS AND COUNTING

Find the indicated sets.

1) List all subsets of the following set. { Al, Bob}	2) List all subsets of the following set. { Al, Bob, Chris}
3) List the elements of the following set. { Al, Bob, Chris, Dave} $\cap$ {Bob, Chris, Dave, Ed}	4) List the elements of the following set. { Al, Bob, Chris, Dave} $\cup$ {Bob, Chris, Dave, Ed}

Problems 5 – 8: Let Universal set U = {a, b, c, d, e, f, g, h, i, j}, sets V = {a, e, i, f, h}, W = { a, c, e, g, i}.

List the members of the following sets.

5) $V \cup W$	6) $V \cap W$
7) $\overline{V \cup W}$	8) $\overline{V} \cap \overline{W}$

Problems 9 – 12: Let Universal set U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
and sets A = {1, 2, 3, 4, 5}, B = { 1, 3, 4, 6}, and C = {2, 4, 6}.

List the members of the following sets.

9) $A \cup B$	10) $A \cap C$
11) $\overline{A \cup B} \cap C$	12) $\overline{A} \cup \overline{B \cap C}$

SECTION 7.1 PROBLEM SET: SETS AND COUNTING

Use Venn Diagrams to find the number of elements in the following sets.

13) In Mrs. Yamamoto's class of 35 students, 12 students are taking history, 18 are taking English, and 4 are taking both. Draw a Venn diagram and use it determine how many students are taking neither history nor English.

14) In a survey of 1200 college students, 700 used Spotify to listen to music and 400 used iTunes to listen to music; of these, 100 used both.
a. Draw a Venn Diagram and find the number of people in each region of the diagram.
b. How many used either Spotify or iTunes?

15) A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, and 15 used both. How many athletes were surveyed?

16) In 2016, 80 college students were surveyed about what video services they subscribed to. Suppose the survey showed that 50 use Amazon Prime, 30 use Netflix, 20 use Hulu. Of those, 13 use Amazon Prime and Netflix, 9 use Amazon Prime and Hulu, 7 use Netflix and Hulu. 3 students use all three services.
a. Draw a Venn Diagram and use it to determine the number of people in each region of the diagram.
b. How many use at least one of these?
c. How many use none of these?

SECTION 7.1 PROBLEM SET: SETS AND COUNTING

17) A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.

a) Find the number of students taking math but not the other two subjects.

b) The number of students taking English or math but not history.

c) The number of students taking none of these subjects.

18) In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.

a) Find the number of investors that participated in the survey.

b) How many invested in stocks or mutual funds but not in bonds?

c) How many invested in exactly one type of investment?

SECTION 7.1 PROBLEM SET: SETS AND COUNTING

19) A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. (This question relates back to question #17.)
For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Students who take at least one of these classes	b. Students who take exactly one of these classes
c. Students who take at least two of these classes	d. Students who take exactly two of these classes
e. Students who take at most two of these classes	f. Students who take English or Math but not both
g. Students who take Math or History but not English	h. Students who take all of these classes

SECTION 7.1 PROBLEM SET: SETS AND COUNTING

20) In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. (This question relates back to question #18.)
For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Investors who invested in mutual funds only	b. Investors who invested in stocks and bonds but not mutual funds
c. Investors who invested in exactly one of these investments	d. Investors who invested in exactly two of these investments
e. Investors who invested in at least two of these investments	f. Investors who invested in at most two of these investments
g. Investors who did not invest in bonds	h. Investors who invested in all three investments.

7.2 Tree Diagrams and the Multiplication Axiom

In this section you will learn to

1. *Use trees to count possible outcomes in a multi-step process*
2. *Use the multiplication axiom to count possible outcomes in a multi-stop process.*

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

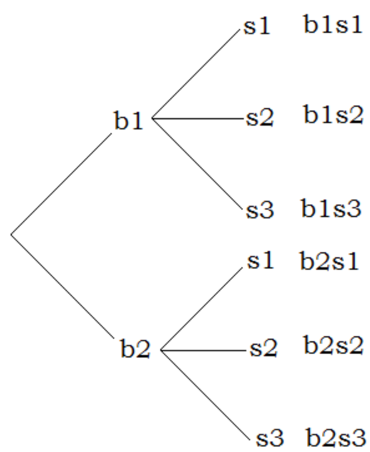
◆***Example 1*** If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?

Solution: Suppose we call the blouses b_1 and b_2, and skirts s_1, s_2, and s_3.

We can have the following six outfits.

$$b_1s_1,\ b_1s_2,\ b_1s_3,\ b_2s_1,\ b_2s_2,\ b_2s_3$$

Alternatively, we can draw a tree diagram:



The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

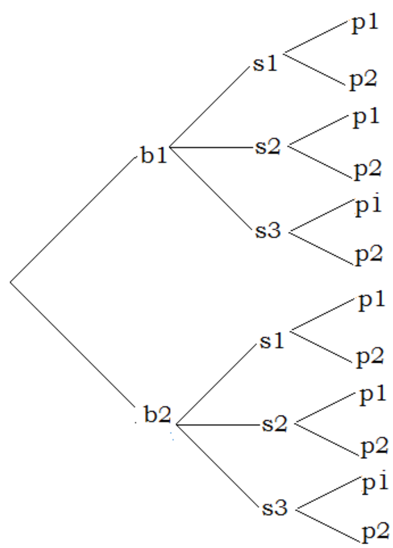
The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ possibilities.

If, in the previous example, we add the shoes to the outfit, we have the following problem.

◆***Example 2*** If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?

Solution: Suppose we call the blouses b_1 and b_2, the skirts s_1, s_2, and s_3, and the pumps p_1, and p_2.

The following tree diagram results.



We count the number of branches in the tree, and see that there are 12 different possibilities.

This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1s_2p_2$. By looking at the different branches on the tree, one can easily see the other possibilities.

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps.

All in all, we have $2 \cdot 3 \cdot 2 = 12$ different possibilities.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example 2 without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

The number of ways of choosing a blouse	The number of ways of choosing a skirt	The number of ways of choosing pumps
2	3	2

The procedure we just employed is called the multiplication axiom.

THE MULTIPLICATION AXIOM

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

◆***Example 3*** A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution: Since there are 26 letters and 10 digits, we have the following choices for each.

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260{,}000$.

◆***Example 4*** In how many different ways can a 3-question true-false test be answered?

Solution: Since there are two choices for each question, we have

Question 1	Question 2	Question 3
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

◆***Example 5*** In how many different ways can four people be seated in a row?

Solution: Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

4	3	2	1

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

◆***Example 6*** How many three-letter word sequences can be formed using the letters {A, B, C} if no letter is to be repeated?

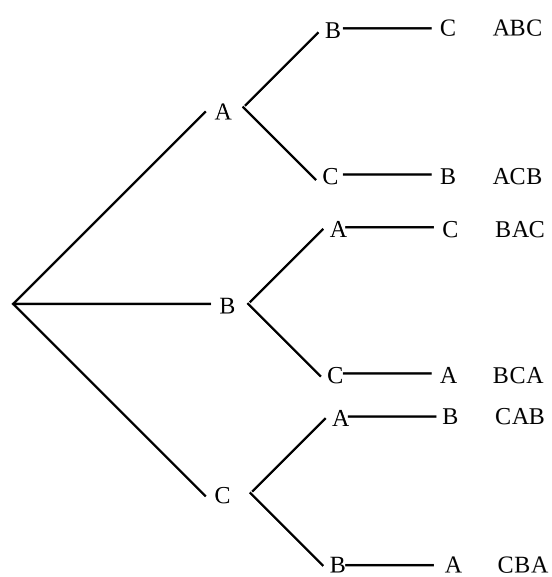
Solution: The problem is very similar to the previous example.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

3	2	1

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram showing all six possibilities.



SECTION 7.2 PROBLEM SET: TREE DIAGRAMS AND THE MULTIPLICATION AXIOM

Do the following problems using a tree diagram or the multiplcation axiom.

1) A man has 3 shirts, and 2 pairs of pants. Use a tree diagram to determine the number of possible outfits.	2) In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices.
3) There are 4 roads from Town A to Town B, 2 roads from Town B to Town C. Use a tree diagram to find the number of ways one can travel from Town A to Town C.	4) Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available.
5) For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink.	6) A California license plate consists of a number from 1 to 5, then three letters followed by three digits. How many such plates are possible?

SECTION 7.2 PROBLEM SET: TREE DIAGRAMS AND THE MULTIPLICATION AXIOM

Do the following problems using the Multiplication Axiom.

7) A license plate consists of three letters followed by three digits. How many license plates are possible if no letter may be repeated?	8) How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated?
9) How many seven-digit telephone numbers are possible if the first two digits cannot be ones or zeros?	10) How many 3-letter word sequences can be formed using the letters {a, b, c, d} if no letter is to be repeated?

Use a tree diagram for questions 11 and 12:

11) A family has two children, use a tree diagram to determine all four possibilities of outcomes by gender.	12) A coin is tossed three times and the sequence of heads and tails is recorded. Use a tree diagram to list all the possible outcomes.

SECTION 7.2 PROBLEM SET: TREE DIAGRAMS AND THE MULTIPLICATION AXIOM

Do the following problems using the Multiplication Axiom.

13) In how many ways can a 4-question true-false test be answered?	14) In how many ways can three people be arranged to stand in a straight line?
15) A combination lock is opened by first turning to the left, then to the right, and then to the left again. If there are 30 digits on the dial, how many possible combinations are there?	16) How many different answers are possible for a multiple-choice test with 10 questions and five possible answers for each question?
17) In the past, a college required students to use a 4 digit PIN (Personal Identification Number) as their password for its registration system. How many different PINs are possible if each must have 4 digits with no restrictions on selection or arrangement of the digits used?	18) The college decided that a more secure password system is needed. New passwords must have 3 numerical digits followed by 6 letters. There are no restrictions on the selection of the numerical digits. However, the letters I and O are not permitted. How many different passwords are possible?

7.3 Permutations

In this section you will learn to

1. *count the number of possible permutations (ordered arrangement) of n items taken r at a time*
2. *count the number of possible permutations when there are conditions imposed on the arrangements*
3. *perform calculations using factorials*

In Example 6 of section 6.2, we were asked to find the word sequences formed by using the letters {A, B, C} if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

◆***Example 1*** How many three-letter word sequences can be formed using the letters {A, B, C, D}?

Solution: There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

4	3	2

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

◆***Example 2*** How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?

Solution: In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4						3

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

◆***Example 3*** Given five letters {A, B, C, D, E}. Find the following:

a. The number of four-letter word sequences.

b. The number of three-letter word sequences.

c. The number of two-letter word sequences.

Solution: The problem is easily solved by the multiplication axiom, and answers are as follows:

a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.

b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$.

c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we write it as **nPr**.

Therefore, the above example can also be answered as listed below.

a. The number of four-letter word sequences is 5P4 = 120.

b. The number of three-letter word sequences is 5P3 = 60.

c. The number of two-letter word sequences is 5P2 = 20.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in the next chapter.

Factorial

$$n! = n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1.$$

where n is a natural number.

$$0! = 1$$

Now we define nPr.

The Number of Permutations of n Objects Taken r at a Time

$$nPr = n(n-1)(n-2)(n-3) \cdots (n-r+1), \text{ or}$$

$$nPr = \frac{n!}{(n-r)!}$$

where n and r are natural numbers.

The reader should become familiar with both formulas and should feel comfortable in applying either.

◆***Example 4*** Compute the following using both formulas.

a. 6P3 b. 7P2

Solution: We will identify n and r in each case and solve using the formulas provided.

a. $6P3 = 6 \cdot 5 \cdot 4 = 120$, alternately

$$6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

b. $7P2 = 7 \cdot 6 = 42$, or

$$7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$$

Next we consider some more permutation problems to get further insight into these concepts.

◆***Example 5*** In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

Solution: Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are [AB] CD. Since [AB] is treated as one person, we have the following possible arrangements.

[AB] CD, [AB] DC, C[AB] D, D[AB] C, CD[AB] , DC[AB]

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

[BA] CD, [BA] DC, C[BA] D, D[BA] C, CD[BA] , DC[BA]

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are 3! 2! = 12 different arrangements

◆***Example 6*** You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

Solution: We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot, 3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$.

Alternately, we can see that $4 \cdot 3 \cdot 2$ is really same as 4P3, and $5 \cdot 4$ is 5P2.

So the answer can be written as (4P3) (5P2) = 480.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are 5P2 permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer (4P3) (5P2).

We summarize the concepts of this section:

1. **Permutations**

   A permutation of a set of elements is an ordered arrangement where each element is used once.

2. **Factorial**

   $n! = n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1.$

   Where n is a natural number.

   $0! = 1$

3. **Permutations of n Objects Taken r at a Time**

   $nPr = n(n-1)(n-2)(n-3) \cdots (n-r+1)$, or

   $$nPr = \frac{n!}{(n-r)!}$$

   where n and r are natural numbers.

SECTION 7.3 PROBLEM SET: PERMUTATIONS

Do the following problems using permutations.

1) How many three-letter words can be made using the letters {a, b, c, d, e} if no repetitions are allowed?	2) A grocery store has five checkout counters, and seven clerks. How many different ways can the 7 clerks be assigned to the 5 counters?
3) A group of fifteen people who are members of an investment club wish to choose a president, and a secretary. How many different ways can this be done?	4) Compute the following. a) 9P2 b) 6P4 c) 8P3 d) 7P4
5) In how many ways can the letters of the word CUPERTINO be arranged if each letter is used only once in each arrangement?	6) How many permutations of the letters of the word PROBLEM end in a vowel?
7) How many permutations of the letters of the word SECURITY end in a consonant?	8) How many permutations of the letters PRODUCT have consonants in the second and third positions?

SECTION 7.3 PROBLEM SET: PERMUTATIONS

Do the following problems using permutations.

9) How many three-digit numbers are there?	10) How many three-digit odd numbers are there?
11) In how many different ways can five people be seated in a row if two of them insist on sitting next to each other?	12) In how many different ways can five people be seated in a row if two of them insist on not sitting next to each other?
13) In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if the English books are set on the left, history books in the middle, and math books on the right?	14) In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if they are grouped by subject?
15) You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with math books and the next three with history books?	16) You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with the books of one subject and the next three slots are to be filled with the books of the other subject?

SECTION 7.3 PROBLEM SET: PERMUTATIONS

Do the following problems using permutations.

17) A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case?	18) A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row?
19) At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program?	20) A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible?
21) In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU?	22) a. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX? b. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other?

7.4 Circular Permutations and Permutations with Similar Elements

In this section you will learn to

1. *count the number of possible permutations of items arranged in a circle*
2. *count the number of possible permutations when there are repeated items*

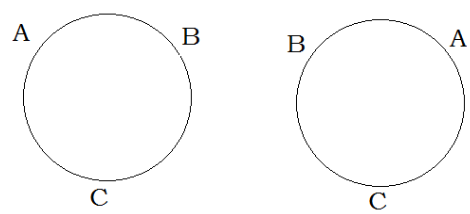
In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

CIRCULAR PERMUTATIONS

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.



It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$

◆***Example 1*** In how many different ways can five people be seated at a circular table?

Solution: We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

1	4	3	2	1

So the answer is 24.

◆***Example 2*** In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

Solution: We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

1	4	3	3	2	2	1	1

So the answer is 144.

PERMUTATIONS WITH SIMILAR ELEMENTS

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$E_1LE_2ME_3NT$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$LE_1ME_2NE_3T$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$LE_1ME_2NE_3T$

$LE_1ME_3NE_2T$

$LE_2ME_1NE_3T$

$LE_2ME_3NE_1T$

$LE_3ME_2NE_1T$

$LE_3ME_1NE_2T$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1LE_2ME_3NT$.

But we know there are 7! permutations of the letters $E_1LE_2ME_3NT$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1!\, r_2! \ldots r_k!}$$

◆***Example 3*** Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution: The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34{,}650$.

◆***Example 4*** If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution: Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!\, 2!} = 15$.

◆***Example 5*** In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution: Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!\, 3!\, 2!} = 1260$$

◆***Example 6*** A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution: This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!\, 5!\, 5!\, 5!} = 11{,}732{,}745{,}024$$

◆***Example 7*** A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

Solution: The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG,

YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

1. Circular Permutations

The number of permutations of n elements in a circle is

$$(n - 1)!$$

2. Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $n = r_1 + r_2 + \cdot\ \cdot\ \cdot + r_k$ is

$$\frac{n!}{r_1!\, r_2! \ldots r_k!}$$

This is also referred to as **ordered partitions**.

SECTION 7.4 PROBLEM SET: CIRCULAR PERMUTATIONS AND PERMUTATIONS WITH SIMILAR ELEMENTS

Do the following problems using the techniques learned in this section.

1) In how many different ways can five children hold hands to play "Ring Around the Rosy"?	2) In how many ways can three people be made to sit at a round table?
3) In how many different ways can six children ride a "Merry Go Around" with six horses?	4) In how many ways can three couples be seated at a round table, so that men and women sit alternately?
5) In how many ways can six trinkets be arranged on a chain?	6) In how many ways can five keys be put on a key ring?
7) Find the number of different permutations of the letters of the word MASSACHUSETTS.	8) Find the number of different permutations of the letters of the word MATHEMATICS.

SECTION 7.4 PROBLEM SET: CIRCULAR PERMUTATIONS AND PERMUTATIONS WITH SIMILAR ELEMENTS

9) Seven flags are to be flown on seven poles: 3 flags are red, 2 are white, and 2 are blue,. How many different arrangements are possible?	10) How many different ways can 3 pennies, 2 nickels and 5 dimes be arranged in a row?
11) How many four-digit numbers can be made using two 2's and two 3's?	12) How many five-digit numbers can be made using two 6's and three 7's?
13) If a coin is tossed 5 times, how many different outcomes of 3 heads and 2 tails are possible?	14) If a coin is tossed 10 times, how many different outcomes of 7 heads and 3 tails are possible?
15) If a team plays ten games, how many different outcomes of 6 wins and 4 losses are possible?	16) If a team plays ten games, how many different ways can the team have a winning season?

7.5 Combinations

In this section you will learn to

1. Count the number of combinations of r out of n items (selections without regard to arrangement)

2. Use factorials to perform calculations involving combinations

Suppose we have a set of three letters {A, B, C}, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people {A, B, C} as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** – the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in the above example, 3P2 = 6, and 3C2 = 3.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

◆***Example 1*** Given the set of letters {A, B, C, D}. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

Solution: We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

ABC	BCD	CDA	BDA
ACB	BDC	CAD	BAD
BAC	CDB	DAC	DAB
BCA	CBD	DCA	DBA
CAB	DCB	ACD	ADB
CBA	DBC	ADC	ABD

The number of permutations are 3! times the number of combinations; that is

$$4P3 = 3! \cdot 4C3$$

or

$$4C3 = \frac{4P3}{3!}$$

In general, $nCr = \dfrac{nPr}{r!}$

Since $nPr = \dfrac{n!}{(n-r)!}$

We have, $nCr = \dfrac{n!}{(n-r)!\, r!}$

Summarizing,

1. Combinations

A combination of a set of elements is an arrangement where each element is used once, and order is not important.

2. The Number of Combinations of n Objects Taken r at a Time

$$nCr = \frac{n!}{(n-r)!\, r!}$$

where n and r are natural numbers.

◆***Example 2*** Compute: a. 5C3 b. 7C3.

Solution: We use the above formula.

$$5C3 = \frac{5!}{(5-3)!\, 3!} = \frac{5!}{2!\, 3!} = 10$$

$$7C3 = \frac{7!}{(7-3)!\, 3!} = \frac{7!}{4!\, 3!} = 35.$$

◆***Example 3*** In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

Solution: Since the order is not important, it is a combination problem, and the answer is

$$7C5 = 21.$$

◆***Example 4*** How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

Solution: Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$6C2 = \frac{6!}{4!\,2!} = 15$$

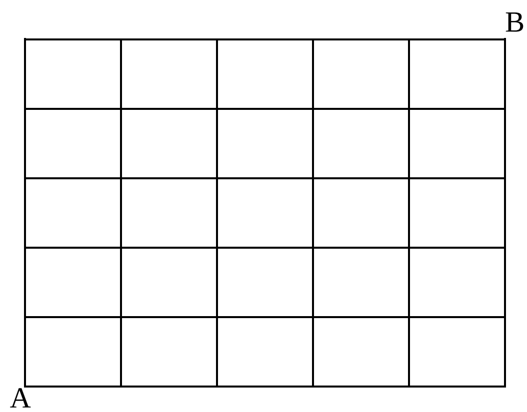
◆***Example 5*** There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

Solution: Note that between any two people there is only one hand shake. Therefore, we have

$$10C2 = 45 \text{ hand-shakes.}$$

◆***Example 6*** The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

Solution: Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.



To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is 10C5, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

HHHHHVVVVV

Clearly there are $\frac{10!}{5!\,5!} = 252$ permutations.

Further note that by definition $10C5 = \frac{10!}{5!\,5!}$.

◆***Example 7*** If a coin is tossed six times, in how many ways can it fall four heads and two tails?

Solution: First we solve this problem using section 6.5 technique–permutations with similar elements.

We need 4 heads and 2 tails, that is

HHHHTT

There are $\frac{6!}{4!\,2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

$6C4 = 15.$

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

$6C2 = 15.$

Further observe that by definition

$$6C4 = \frac{6!}{2!\,4!}$$

and $$6C2 = \frac{6!}{4!\,2!}$$

Which implies $6C4 = 6C2$.

SECTION 7.5 PROBLEM SET: COMBINATIONS

Do the following problems using combinations.

1) How many different 3-people committees can be chosen from ten people?	2) How many different 5-player teams can be chosen from eight players?
3) In how many ways can a person choose to vote for three out of five candidates on a ballot for a school board election?	4) Compute the following: a) 9C2 b) 6C4 c) 8C3 d) 7C4
5) How many 5-card hands can be chosen from a deck of cards?	6) How many 13-card bridge hands can be chosen from a deck of cards?
7) There are twelve people at a party. If they all shake hands, how many different hand-shakes are there?	8) In how many ways can a student choose to do four questions out of five on a test?

SECTION 7.5 PROBLEM SET: COMBINATIONS

Do the following problems using combinations.

9) Five points lie on a circle. How many chords can be drawn through them?	10) How many diagonals does a hexagon have?
11) There are five teams in a league. How many games are played if every team plays each other twice?	12) A team plays 15 games a season. In how many ways can it have 8 wins and 7 losses?
13) In how many different ways can a 4-child family have 2 boys and 2 girls?	14) A coin is tossed five times. In how many ways can it fall three heads and two tails?
15) The shopping area of a town is a square that is six blocks by six blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?	16) If the shopping area in the previous problem has a rectangular form of 5 blocks by 3 blocks, then how many different routes can a taxi driver take to drive from one end of the shopping area to the opposite kitty corner end?

SECTION 7.5 PROBLEM SET: COMBINATIONS

Do the following problems using combinations.

17) A team of 7 workers is assigned to a project. In how many ways can 3 of the 7 workers be selected to make a presentation to the management about their progress on the project?	18) A real estate company has 12 houses listed for sale by their clients. In how many ways can 5 of the 12 houses be selected to be featured in an advertising brochures?
19) A frozen yogurt store has 9 toppings to choose from. In how many ways can 3 of the 9 toppings be selected ?	20) A kindergarten teacher has 14 books about a holiday. In how many ways can she select 4 of the books to read to her class in the week before the holiday?

7.6 Combinations: Involving Several Sets

In this section you will learn to

1. count the number of items selected from more than one set

2. count the number of items selected when there are restrictions on the selections

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

◆***Example 1*** How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

Solution: We list 4 men and 4 women as follows:

$$M_1M_2M_3M_4W_1W_2W_3W_4$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

2-Man Committees	3-Woman Committees
M_1M_2	$W_1W_2W_3$
M_1M_3	$W_1W_2W_4$
M_1M_4	$W_1W_3W_4$
M_2M_3	$W_2W_3W_4$
M_2M_4	
M_3M_4	

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose M_1M_2 as our 2-man committee, then we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1M_2}\ W_1W_2W_3,\ \boxed{M_1M_2}\ W_1W_2W_4,\ \boxed{M_1M_2}\ W_1W_3W_4,\ \boxed{M_1M_2}\ W_2W_3W_4$$

Similarly, if we choose M_1M_3 as our 2-man committee, then, again, we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees.

$$\boxed{M_1M_3}\ W_1W_2W_3,\ \boxed{M_1M_3}\ W_1W_2W_4,\ \boxed{M_1M_3}\ W_1W_3W_4,\ \boxed{M_1M_3}\ W_2W_3W_4$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3-woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

◆***Example 2*** A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?
b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

Solution: a. Applying the multiplication axiom to the combinations involved, we get

(4C1) (5C1) (5C1) (6C1) = 600

b. We are choosing all 4 members from the 5 juniors, and none from the others.

5C4 = 5

c. 4C2 · 6C2 = 90

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

16C4 = 1820

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

(6C3) (14C1) + 6C4 = 295

◆***Example 3*** How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

Solution: First we select a group of five letters consisting of 2 vowels and 3 consonants. Since there are 4 vowels and 5 consonants, we have

(4C2) (5C3)

Since our next task is to make word sequences out of these letters, we multiply these by 5!.

(4C2) (5C3) (5!) = 7200.

◆***Example 4*** A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?

Solution: We will do the problem using the following steps.

Step 1. Select a suit.
Step 2. Select four cards from this suit.
Step 3. Select another suit.
Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

Ways of selecting the first suit	Ways of selecting 4 cards from this suit	Ways of selecting the next suit	Ways of selecting a card from that suit
4C1	13C4	3C1	13C1

(4C1) (13C4) (3C1) (13C1) = 111,540.

A STANDARD DECK OF 52 PLAYING CARDS

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit.

♦ diamonds ♥ hearts ♠ spades ♣ clubs

Each suit is associated with a color, either black (spades ,clubs) or red (diamonds ,hearts)

Each suit contains 13 denominations (or values) for cards:

nine numbers 2, 3, 4, …., 10 and Jack(J), Queen (Q), King (K), Ace (A).

The Jack, Queen and King are called "face cards" because they have pictures on them.
Therefore a standard deck has12 face cards: (3 values JQK) x (4 suits ♦ ♥ ♠ ♣)

We can visualize the 52 cards by the following display

Suit	Color	Values (Denominations)
♦ Diamonds	Red	2 3 4 5 6 7 8 9 10 J Q K A
♥ Hearts	Red	2 3 4 5 6 7 8 9 10 J Q K A
♠ Spades	Black	2 3 4 5 6 7 8 9 10 J Q K A
♣ Clubs	Black	2 3 4 5 6 7 8 9 10 J Q K A

SECTION 7.6 PROBLEM SET: COMBINATIONS INVOLVING SEVERAL SETS

Following problems involve combinations from several different sets.

1) How many 5-people committees consisting of three boys and two girls can be chosen from a group of four boys and four girls?	2) A club has 4 men, 5 women, 8 boys and 10 girls as members. In how many ways can a group of 2 men, 3 women, 4 boys and 4 girls be chosen?
3) How many 4-people committees chosen from 4 men and 6 women will have at least 3 men?	4) A batch contains 10 transistors of which three are defective. If three are chosen, in how many ways can they be selected with two defective?
5) In how many ways can five counters labeled A, B, C, D and E at a store be staffed by two men and three women chosen from a group of four men and six women?	6) How many 4-letter word sequences consisting of two vowels and two consonants can be made from the letters of the word PHOENIX if no letter is repeated?

Three marbles are chosen from an urn that contains 5 red, 4 white, and 3 blue marbles.
How many samples of the following type are possible?

7) All three white.	8) Two blue and one white.
9) One of each color.	10) All three of the same color.
11) At least two red.	12) None red.

SECTION 7.6 PROBLEM SET: COMBINATIONS INVOLVING SEVERAL SETS

Following problems involve combinations from several different sets.

Five coins are chosen from a bag that contains 4 dimes, 5 nickels, and 6 pennies. How many samples of five coins of the following types are possible?

13) At least four nickels.	14) No pennies.
15) Five of a kind.	16) Four of a kind.
17) Two of one kind and two of another kind.	18) Three of one kind and two of another kind.

Find the number of different ways to draw a 5-card hand from a deck to have the following combinations.

19) Three face cards.	20) A heart flush (all hearts).
21) Two hearts and three diamonds.	22) Two cards of one suit, and three of another suit.
23) Two kings and three queens.	24) 2 cards of one value and 3 of another value.

SECTION 7.6 PROBLEM SET: COMBINATIONS INVOLVING SEVERAL SETS

The party affiliation of the 100 United States Senators in the 114^{th} Congress, January 2015, was:
44 Democrats, 54 Republicans, and 2 Independents.

25) In how many ways could a 10 person committee be selected if it is to contain 4 Democrats, 5 Republicans, and 1 Independent?	26) In how many different ways could a 10 person committee be selected with 6 or 7 Republicans and the rest Democrats (with no Independents)?

The 100 United States Senators in the 114^{th} Congress, January 2015, included 80 men and 20 women. Suppose a committee senators is working on legislation about wage discrimination by gender.

27) In how many ways could a 12 person committee be selected to contain equal numbers of men and women.	28) In how many ways could a 6 person committee be selected to contain fewer women than men?

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

How many different playlists of 6 songs can be selected that satisfy each of the following:
(We care which songs are selected to be on the playlist, but not what order they are selected or listed in.)

29) Playlist has 2 songs of each type	30) Playlist has no country songs
31) Playlist has 3 rock, 2 rap, and 1 country song	32) Playlist has 3 or 4 rock songs and all the rest are rap songs

7.7 Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^2y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^6 = x6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

We make the following observations.

1. There are $n + 1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term. The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3 + \square x^2y + \square xy^2 + \square y^3 \qquad \text{(I)}$$

Our next job is to replace each of the blanks in equation (I) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors (x+y)(x+y)(x+y). For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (l), and we get

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

♦ ***Example 1*** Find the coefficient of the term x^2y^5 in the expansion $(x + y)^7$.

Solution: The expansion $(x + y)^7 = (x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)$

In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)$.

The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term x^2y^5 is 21.

♦ ***Example 2*** Expand $(x + y)^7$

Solution: We first write the expansion without the coefficients.

$$(x + y)^7 = \square x^7 + \square x^6y + \square x^5y^2 + \square x^4y^3 + \square x^3y^4 + \square x^2y^5 + \square xy^6 + \square y^7$$

Now we determine the coefficient of each term as we did in Example 1.

The coefficient of lhe term x^7 is 7C7 or 7CO which equals 1.

The coefficient of the term x^6y is 7C6 or 7C1 which equals 7.

The coefficient of the term x^5y^2 is 7C5 or 7C2 which equals 21.

The coefficient of the term x^4y^3 is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7 = x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+ 7xy^6+ y^7$

We generalize the result.

Binomial Theorem
$(x + y)^n = {}_nC_0\, x^n + {}_nC_l\, x^{n-1} y + {}_nC_2\, x^{n-2} y^2 + \cdots + {}_nC_{n-1}\, xy^{n-1} + {}_nC_n\, y^n$

♦ ***Example 3*** Expand $(3a–2b)^4$

Solution: If we let x = 3a and y = – 2b, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4C0(3a)^4 + 4Cl(3a)^3(-2b) + 4C2(3a)^2(-2b)^2 + 4C3(3a)(-2b)^3 + 4C4(-2b)^4\\ &= 1(8la^4) + 4(27a^3)(-2b) + 6(9a^2)(4b^2) + 4(3a)(-8b^3) + 1(16b^3)\\ &= 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4\end{aligned}$$

♦ ***Example 4*** Find the fifth term of the expansion $(3a–2b)^7$.

Solution: The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the y term is always one less than r, and, the coefficient of the term is ${}_nC_{r-1}$.

n = 7 and r – 1 = 5 – 1= 4, so the coefficient is 7C4 = 35

Thus, the fifth term is $(7C4)(3a)^3(-2b)^4 = 35(27a^3)(16b^4) = 15120\, a^3 b^4$

SECTION 7.7 PROBLEM SET: BINOMIAL THEOREM

Use the Binomial Theorem to do the following problems.

1) Expand $(a + b)^5$.	2) Expand $(a - b)^6$.
3) Expand $(x - 2y)^5$.	4) Expand $(2x - 3y)^4$.
5) Find the third term of $(2x - 3y)^6$.	6) Find the sixth term of $(5x + y)^8$.

SECTION 7.7 PROBLEM SET: BINOMIAL THEOREM

Use the Binomial Theorem to do the following problems.

7) Find the coefficient of the x^3y^4 term in the expansion of $(2x + y)^7$.	8) Find the coefficient of the a^4b^6 term in the expansion of $(3a - b)^{10}$.
9) A coin is tossed 5 times, in how many ways is it possible to get three heads and two tails?	10) A coin is tossed 10 times, in how many ways is it possible to get seven heads and three tails?
11) How many subsets are there of a set that has 6 elements?	12) How many subsets are there of a set that has n elements?

SECTION 7.8 PROBLEM SET: CHAPTER REVIEW

1) Suppose of the 4,000 freshmen at a college everyone is enrolled in a mathematics or an English class during a given quarter. If 2,000 are enrolled in a mathematics class, and 3,000 in an English class, how many are enrolled in both a mathematics class and an English class?

2) In a survey of 250 people, it was found that 125 had read Time magazine, 175 had read Newsweek, 100 had read U. S. News, 75 had read Time and Newsweek, 60 had read Newsweek and U. S. News, 55 had read Time and U. S. News, and 25 had read all three.

   a) How many had read Time but not the other two?

   b) How many had read Time or Newsweek but not the U. S. News And World Report?

   c) How many had read none of these three magazines?

3) At a manufacturing plant, a product goes through assembly, testing, and packing. If a plant has three assembly stations, two testing stations, and two packing stations, in how many different ways can a product achieve its completion?

4) Six people are to line up for a photograph. How many different lineups are possible if three of them insist on standing next to each other ?

5) How many four-letter word sequences can be made from the letters of the word CUPERTINO?

6) In how many different ways can a 20-question multiple choice test be designed so that its answers contain 2 A's, 4 B's, 9 C's, 3 D's, and 2 E's?

7) The U. S. Supreme Court has nine judges. In how many different ways can the judges cast a six-to-three decision in favor of a ruling?

8) In how many different ways can a coach choose a linebacker, a guard, and a tackle from five players on the bench, if all five can play any of the three positions?

9) How many three digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if no repetitions are allowed?

10) Compute: a) 9C4 b) 8P3 c) $\frac{10!}{4!\,(10-4)!}$

11) In how many ways can 3 English, 3 Math, and 4 Spanish books be set on a shelf if the books are grouped by subject?

12) In how many ways can a 10-question multiple choice test with four possible answers for each question be answered?

13) On a soccer team three fullbacks can play any of the three fullback positions, left, center, and right. The three halfbacks can play any of the three halfback positions, the four forwards can play any of the four positions, and the goalkeeper plays only his position. How many different arrangements of the 11 players are possible?

14) From a group of 6 people, 3 are assigned to cleaning, 2 to hauling and one to garbage collecting. How many different ways can this be done?

15) How many three-letter word sequences can be made from the letters of the word OXYGEN?

SECTION 7.8 PROBLEM SET: CHAPTER REVIEW

16) In how many ways can 3 books be selected from 4 English and 2 History books if at least one English book must be chosen?

17) Five points lie on the rim of a circle. Choosing the points as vertices, how many different triangles can be drawn?

18) A club consists of six men and nine women. In how many ways can a president, a vice president and a treasurer be chosen if the two of the officers must be women?

19) Of its 12 sales people, a company wants to assign 4 to its Western territory, 5 to its Northern territory, and 3 to its Southern territory. How many ways can this be done?

20) How many permutations of the letters of the word OUTSIDE have consonants in the first and last place?

21) How many distinguishable permutations are there in the word COMMUNICATION?

22) How many five-card poker hands consisting of the following distribution are there?
   a) A flush(all five cards of a single suit)
   b) Three of a kind(e.g. three aces and two other cards)
   c) Two pairs(e.g. two aces, two kings and one other card)
   d) A straight(all five cards in a sequence)

23) Company stocks on an exchange are given symbols consisting of three letters. How many different three-letter symbols are possible?

24) How many four-digit odd numbers are there?

25) In how many ways can 7 people be made to stand in a straight line? In a circle?

26) A United Nations delegation consists of 6 Americans, 5 Russians, and 4 Chinese. Answer the following questions.
   a) How many committees of five people are there?
   b) How many committees of three people consisting of at least one American are there?
   c) How many committees of four people having no Russians are there?
   d) How many committees of three people have more Americans than Russians?
   e) How many committees of three people do not have all three Americans?

27) If a coin is flipped five times, in how many different ways can it show up three heads?

28) To reach his destination, a man is to walk three blocks north and four blocks west. How many different routes are possible?

29) All three players of the women's beach volleyball team, and all three players of the men's beach volleyball team are to line up for a picture with all members of the women's team lined together and all members of men's team lined up together. How many ways can this be done?

30) From a group of 6 Americans, 5 Japanese and 4 German delegates, two Americans, two Japanese and a German are chosen to line up for a photograph. In how many different ways can this be done?

31) Find the fourth term of the expansion $(2x - 3y)^8$.

32) Find the coefficient of the a^5b^4 term in the expansion of $(a – 2b)^9$.

CHAPTER 8

Chapter 8: Probability

In this chapter, you will learn to:

1. *Write sample spaces.*
2. *Determine whether two events are mutually exclusive.*
3. *Use the Addition Rule.*
4. *Calculate probabilities using both tree diagrams and combinations.*
5. *Do problems involving conditional probability.*
6. *Determine whether two events are independent.*

8.1 Sample Spaces and Probability

In this section, you will learn to:

1. *Write sample spaces.*
2. *Calculate probabilities by examining simple events in sample spaces*

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel.

It is for this reason, we emphasize the need for understanding sample spaces.

SAMPLE SPACES

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**.

A **sample space** of an experiment is the set of all possible outcomes.

◆***Example 1*** If a die is rolled, write a sample space.

Solution: A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes S is

{1, 2, 3, 4, 5, 6}.

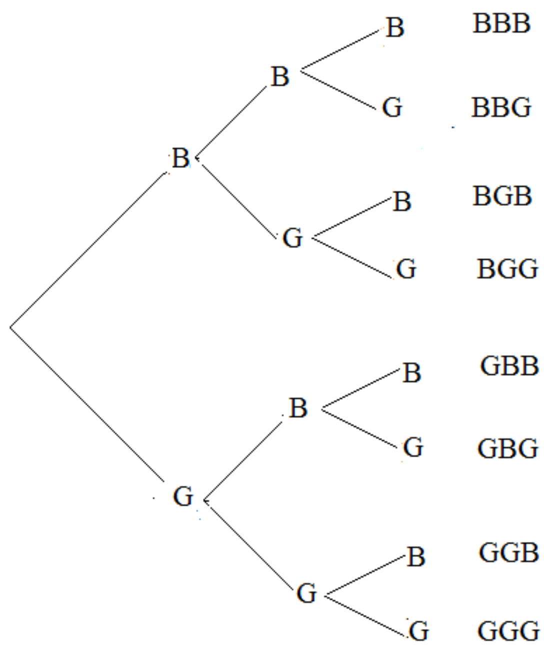
◆***Example 2*** A family has three children. Write a sample space.

Solution: The sample space consists of eight possibilities.

{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.



◆***Example 3*** Two dice are rolled. Write the sample space.

Solution: We assume one of the dice is red, and the other green. We have the following 36 possibilities.

	Green					
Red	**1**	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

PROBABILITY

Now that we understand the concept of a sample space, we will define probability.

> **Probability**
>
> For a sample space S, and an outcome A of S, the following two properties are satisfied.
>
> 1. If A is an outcome of a sample space, then the probability of A, denoted by P(A), is between 0 and 1, inclusive.
>
> $$0 \le P(A) \le 1$$
>
> 2. The sum of the probabilities of all the outcomes in S equals 1.

The probability P(A) of an event A describes the chance or likelihood of that event occurring.

If P(A) = 0, event A is certain not to occur. If P(A) = 1, event A is certain to occur.

If P(A) = 0.5, then event A is equally likely to occur or not occur.

If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50. If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

◆***Example 4*** If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

Solution: Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example 3, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space. We are often interested in finding probabilities of several outcomes represented by an event.

An **event** is a subset of a sample space. If an event consists of only one outcome, it is called a **simple event.**

◆***Example 5*** If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

Solution: Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2), and (6, 1), so event E is

E = {(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)}

The probability of the event E is

P(E) = 6/36 or 1/6.

◆***Example 6*** A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

Solution: We assume the marbles are r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3. Let the event C represent that the marble is red or blue.

The sample space S = {r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3}

And the event C = {r_1, r_2, r_3, b_1, b_2, b_3}

Therefore, the probability of C,

P(C) = 6/10 or 3/5.

◆***Example 7*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar, and therefore is no longer available to select on the second draw.

Solution: Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

S = {(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)}

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event E represent that the sum of the numbers is five. Then

E = {(2, 3), (3, 2)}

Therefore, the probability of F is

P(E) = 2/6 or 1/3.

◆***Example 8*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution: The sample space, as in Example 7, consists of the following six possibilities.

S = {(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)}

Let the event F represent that the sum of the numbers is at least four. Then

F = {(1, 3), (3, 1), (2, 3), (3, 2)}

Therefore, the probability of F is

P(F) = 4/6 or 2/3.

◆***Example 9*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

Solution: When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

S = {(1,1), (1, 2), (1, 3), (2, 1), (2,2), (2, 3), (3, 1), (3, 2), (3,3)}

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

E = {(2, 3), (3, 2) }

Therefore, the probability of F is P(E) = 2/9

Note that in Example 9 when we selected marbles with replacement, the probability has changed from Example 7 where we selected marbles without replacement.

◆***Example 10*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution: The sample space when drawing with replacement consists of the following nine possibilities.

S = {(1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3,3)}

Let the event F represent that the sum of the numbers is at least four. Then

F = {(1, 3), (3, 1), (2, 3), (3, 2), (2,2), (3,3)}

Therefore, the probability of F is

P(F) = 6/9 or 2/3.

Note that in Example 10 when we selected marbles with replacement, the probability is the same as in Example 8 where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 8.3.

◆***Example 11*** One 6 sided die is rolled once. Find the probability that the result is greater than 4.

Solution: The sample space consists of the following six possibilities in set S: S={1,2,3,4,5,6}

Let E be the event that the number rolled is greater than four: E={5,6}

Therefore, the probability of E is: P(E) = 2/6 or 1/3.

SECTION 8.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

In problems 1 - 6, write a sample space for the given experiment.

1) A die is rolled.	2) A penny and a nickel are tossed.
3) A die is rolled, and a coin is tossed.	4) Three coins are tossed.
5) Two dice are rolled.	6) A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn.

In problems 7 - 12, one card is randomly selected from a deck. Find the following probabilities.

7) P(an ace)	8) P(a red card)
9) P(a club)	10) P(a face card)
11) P(a jack or a spade)	12) P(a jack and a spade)

SECTION 8.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

For problems 13 – 16: A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

13) P(red)	14) P(white)
15) P(red or blue)	16) P(red and blue)

For problems 17 – 22: Consider a family of three children. Find the following probabilities.

17) P(two boys and a girl)	18) P(at least one boy)
19) P(children of both sexes)	20) P(at most one girl)
21) P(first and third children are male)	22) P(all children are of the same gender)

SECTION 8.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

For problems 23 – 27: Two dice are rolled. Find the following probabilities.

23) P(the sum of the dice is 5)	24) P(the sum of the dice is 8)
25) P(the sum is 3 or 6)	26) P(the sum is more than 10)
27) P(the result is a double) (Hint: a double means that both dice show the same value)	

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

28) P(the sum of the numbers is 5)	29) P(the sum of the numbers is odd)
30) P(the sum of the numbers is 9)	31) P(one of the numbers is 3)

For problems 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

32) P(the sum of the numbers is 5)	33) P(the sum of the numbers is 2)

8.2 Mutually Exclusive Events and the Addition Rule

In this section, you will learn to:

1. *Define compound events using union, intersection, and complement.*
2. *Identify mutually exclusive events*
3. *Use the Addition Rule to calculate probability for unions of events.*

In the last chapter, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

The **union** of two events E and F, $E \cup F$, is the set of outcomes that are in E or in F or in both.

The **intersection** of two events E and F, $E \cap F$, is the set of outcomes that are in both E and F.

The **complement** of an event E, denoted by E^C, is the set of outcomes in the sample space S that are not in E. It is worth noting that $P(E^C) = 1 - P(E)$. This follows from the fact that if the sample space has n elements and E has k elements, then E^C has n – k elements. Therefore,

$$P(E^C) = \frac{n-k}{n} = 1 - \frac{k}{n} = 1 - P(E).$$

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: $E \cap F = \varnothing$.

Next we'll determine whether a given pair of events are mutually exclusive.

◆***Example 1*** A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

Solution: Clearly the ace of hearts belongs to both sets. That is

$E \cap F$ = {Ace of hearts} $\neq \varnothing$.

Therefore, the events E and F are not mutually exclusive.

◆***Example 2*** Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

Solution: For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, $G \cap H$ = {(2, 4), (4, 2)} $\neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

◆***Example 3*** A family has three children. Determine whether the following pair of events are mutually exclusive.

M = {The family has at least one boy} N = {The family has all girls}

Solution: Although the answer may be clear, we list both the sets.

M = {BBB, BBG, BGB, BGG, GBB, GBG, GGB} and N = {GGG}

Clearly, $M \cap N = \varnothing$

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

◆***Example 4*** If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution: Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = {1, 2, 3, 4, 5, 6}. The event E = {2, 4, 6}, and event F = {5, 6}

We need to find $P(E \cup F)$.

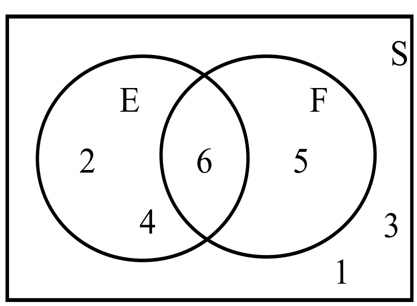
Since P(E) = 3/6, and P(F) = 2/6, a student may say $P(E \cup F)$ = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set $E \cup F$ has only four elements and not five: set $E \cup F$ = {2,4,5,6}

Therefore, $P(E \cup F)$ = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example 4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and $E \cap F$ are listed below.

S = {1, 2, 3, 4, 5, 6}, E = {2, 4, 6}, F = {5, 6}, and $E \cap F$ = {6}.



The above figure shows S, E, F, and $E \cap F$.

Finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number $n(E \cap F)$, we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example 4, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added $P(E \cap F)$ twice. Therefore, we must subtract $P(E \cap F)$, once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event $E \cup F$ is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.**It states

> **Addition Rule:**
>
> $\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$
>
> If, and only if, two events E and F are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

◆***Example 5*** If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

Solution: Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

$$P(A) = 4/52 \text{ and } P(H) = 13/52.$$

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

$$P(A \cap H) = 1/52$$

We need to find $P(A \cup H)$:

$$P(A \cup H) = P(A) + P(H) - P(A \cap H)$$
$$= 4/52 + 13/52 - 1/52 = 16/52.$$

◆***Example 6*** Two dice are rolled, and the events F and T are as follows:
F = {The sum of the dice is four} and T = {At least one die shows a three}
Find $P(F \cup T)$.

Solution: We list F and T, and $F \cap T$ as follows:

F = {(1, 3), (2, 2), (3, 1)}
T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}
$F \cap T$ = {(1, 3), (3, 1)}

Since $P(F \cup T) = P(F) + P(T) - P(F \cap T)$

We have $P(F \cup T) = 3/36 + 11/36 - 2/36 = 12/36$.

◆***Example 7*** Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

Solution: Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

$$P(A) = .80, \quad P(S) = .70, \text{ and } P(A \cup S) = .90.$$

We need to find, $P(A \cap S)$.

The addition formula states that,

$$P(A \cup S) = P(A) + P(S) - P(A \cap S)$$

Substituting the known values, we get

$$.90 = .80 + .70 - P(A \cap S)$$

Therefore, $P(A \cap S) = .60$.

◆***Example 8*** The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

Solution: Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$P(C) = .6, \qquad P(R) = .7, \qquad P(C \cap R) = .5$$

First we find $P(C \cup R)$ using the Addition Rule.

$$P(C \cup R) = P(C) + P(R) - P(C \cap R) = .6 + .7 - .5 = .8$$

Then we find $P((C \cup R)^c)$ using the Complement Rule.

$$P((C \cup R)^c) = 1 - P(C \cup R) = 1 - .8 = .2$$

We summarize this section by listing the important rules.

The Addition Rule

For Two Events E and F, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

The Addition Rule for Mutually Exclusive Events

If Two Events E and F are Mutually Exclusive, then $P(E \cup F) = P(E) + P(F)$

The Complement Rule

If E^c is the Complement of Event E, then $P(E^c) = 1 - P(E)$

SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Determine whether the following pair of events are mutually exclusive.

1) A = {A person earns more than $25,000} B = {A person earns less than $20,000}	2) A card is drawn from a deck. C = {It is a King} D = {It is a heart}.
3) A die is rolled. E = {An even number shows} F = {A number greater than 3 shows}	4) Two dice are rolled. G = {The sum of dice is 8} H = {One die shows a 6}
5) Three coins are tossed. I = {Two heads come up} J = {At least one tail comes up}	6) A family has three children. K = {First born is a boy} L = {The family has children of both sexes}

Use the Addition Rule to find the following probabilities.

7) A card is drawn from a deck. Events C and D are: C = {It is a king} D = {It is a heart} Find P(C or D).	8) A die is rolled. The events E and F are: E = {An even number shows} F = {A number greater than 3 shows} Find P(E or F).
9) Two dice are rolled. Events G and H are: G = {The sum of dice is 8} H ={Exactly one die shows a 6} Find P(G or H).	10) Three coins are tossed> Events I and J are: I = {Two heads come up} J = {At least one tail comes up} Find P(I or J).

SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Use the Addition Rule to find the following probabilities.

11) At a college, 20% of the students take Finite Mathematics, 30% take Statistics and 10% take both. What percent of students take Finite Mathematics or Statistics?	12) This quarter, there is a 50% chance that Jason will pass Accounting, a 60% chance that he will pass English, and 80% chance that he will pass at least one of these two courses. What is the probability that he will pass both Accounting and English?

Questions 13 – 20 refer to the following: The table shows the distribution of Democratic and Republican U.S by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities.

13) P(M and D)	14) P(F and R)
15) P(M or D)	16) P(F or R)
17) P(M^C or R)	18) P(M or F)
19) Are the events F, R mutually exclusive? Use probabilities to support your conclusions.	20) Are the events F, T mutually exclusive? Use probabilities to support your conclusion.

SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Use the Addition Rule to find the following probabilities.

21) If P(E) = .5 , P(F) = .4 , E and F are mutually exclusive, find P(E and F).	22) If P(E) = .4 , P(F) = .2 , E and F are mutually exclusive, find P(E or F).
23) If P(E) = .3, P(E or F) = .6 , P(E and F) = .2, find P(F).	24) If P(E) = .4, P(F) = .5 , P(E or F) = .7, find P(E and F).

25) In a box of assorted cookies, 36% of cookies contain chocolate and 12% of cookies contain nuts. 8% of cookies have both chocolats and nuts. Sean is allergic to chocolate and nuts. Find the probability that a cookie has chocolate chips **or** nuts (he can't eat it).	26) At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper. Find the probability that a course requires a final exam **or** a research paper.

8.3 Probability Using Tree Diagrams and Combinations

In this section, you will learn to:

1. *Use probability tree diagrams to calculate probabilities*
2. *Use combinations to calculate probabilities*

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree. By multiplying probabilities along a path through the tree, we can find probabilities for "and" events, which are intersections of events.

We begin with an example.

◆***Example 1*** Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

Solution: Let E be the event that the first marble drawn is red, and let F be the event that the second marble drawn is red.

We need to find $P(E \cap F)$.

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 49 ordered pairs. Of the 49 ordered pairs, there are $3 \times 3 = 9$ ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F) = \frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F) = \frac{9}{49} = \frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example: $\mathbf{P(E \cap F) = P(E) \cdot P(F)}$

◆***Example 2*** If in Example 1, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

Solution: By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find $P(E \cap F)$.

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are $3 \times 2 = 6$ ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

Here 3/7 represents P(E), and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as P(red on the second | red on first) or P(F | E). The "|" represents the word "given" or "if". This leads to the result that:

$$\mathbf{P(E \cap F) = P(E) \cdot P(F \mid E)}$$

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

◆***Example 3*** Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

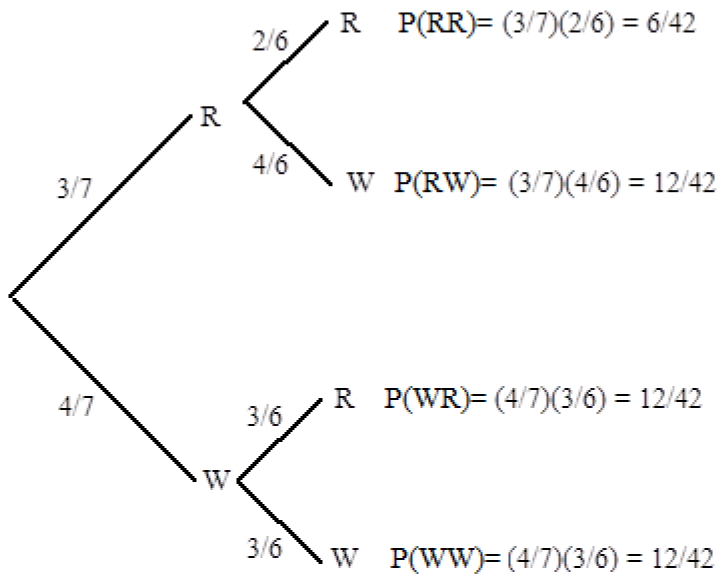
a. The probability that both marbles are red.

b. The probability that the first marble is red and the second white.

c. The probability that one marble is red and the other white.

Solution: Let R be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.



a. The probability that both marbles are red is P(RR)=6/42

b. The probability that the first marble is red and the second is white is

P(RW)=12/42

c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The "or" tells us we'll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

= P(RW or WR)

= P(RW) + P(WR)

= 12/42 + 12/42 = 24/42.

USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

◆***Example 4*** Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white)

b. P(One of each color)

c. P(None blue)

d. P(At least one blue)

Solution: Let us suppose the marbles are labeled as $R_1,R_2,R_3,W_1,W_2,B_1,B_2,B_3$.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are 3C2× 2C1 = 6 combinations consisting of 2 red and one white. Therefore,

$$\text{P(Two red and one white)} = \frac{3C2 \times 2C1}{8C3} = \frac{6}{56}.$$

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are 3C1× 2C1× 3C1 = 18 combinations consisting of one red, one white, and one blue. Therefore,

$$\text{P(One of each color)} = \frac{3C1 \times 2C1 \times 3C1}{8C3} = \frac{18}{56}.$$

c. P(None blue)

There are 5 non-blue marbles, therefore

$$\text{P(None blue)} = \frac{5C3}{8C3} = \frac{10}{56} = \frac{5}{28}.$$

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

P(At least one blue) = P(1 blue, 2 non-blue) + P(2 blue, 1e non-blue) + P(3 blue)

$$\text{P(At least one blue)} = \frac{3C1 \times 5C2}{8C3} + \frac{3C2 \times 5C1}{8C3} + \frac{3C3}{8C3}$$

P(At least one blue) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28.

Alternately, we can use the fact that $P(E) = 1 - P(E^c)$. If the event E = At least one blue, then E^c = None blue.

But from part c of this example, we have $(E^c) = 5/28$, so $P(E) = 1 - 5/28 = 23/28$.

◆***Example 5*** Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

Solution: Let us first do an easier problem–the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$P(\text{A pair of kings and queens}) = \frac{4C2 \times 4C2 \times 44C1}{52C5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are 13C2 different combinations of pairs.

$$P(\text{Two pairs}) = 13C2 \cdot \frac{4C2 \times 4C2 \times 44C1}{52C5} = .04754$$

◆***Example 6*** A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment.
Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

Solution: There are 8C2 ways of selecting 2 out of the 8 iPhones.

and 7C4 ways of selecting 4 out of the 7 Android phones

But altogether there are 15C6 ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(\text{2 iPhones and 4 Android phones}) = \frac{8C2 \times 7C4}{15C6} = \frac{(28)(35)}{5005} = \frac{980}{5005} = 0.1958$$

◆***Example 7*** One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

Solution: There are 20C4 ways of selecting 4 out of the 20 plain bagels,

and 15C3 ways of selecting 3 out of the 15 poppyseed bagels,

and 18C2 ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are 53C9 ways of selecting 9 out of the 53 bagels.

$$P(\text{4 plain, 3 poppyseed, and 2 sesame seed}) = \frac{20C4 \times 15C3 \times 18C2}{53C9}$$

$$= \frac{(4845)(455)(153)}{4431613550}$$

$$= 0.761$$

We end the section by solving a famous problem called the **Birthday Problem**.

◆***Example 8*** If there are 25 people in a room, what is the probability that at least two people have the same birthday?

Solution: Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are 365^{25} possible birthdays for 25 people. Therefore, the sample space has 365^{25} elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$\text{P(No two have the same birthday)} = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365P25}{365^{25}}$$

Since P(at least two people have the same birthday) = 1 – P(No two have the same birthday),

$$\text{P(at least two people have the same birthday)} = 1 - \frac{365P25}{365^{25}} = .5687$$

SECTION 8.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Two apples are chosen from a basket containing five red and three yellow apples.
Draw a tree diagram below, and find the following probabilities.

1) P(both red)	2) P(one red, one yellow)
3) P(both yellow)	4) P(First red and second yellow)

A basket contains six red and four blue marbles. Three marbles are drawn at random.
Find the following probabilities using the method shown in Example 2. Do not use combinations.

5) P(All three red)	6) P(two red, one blue)
7) P(one red, two blue)	8) P(first red, second blue, third red)

Three marbles are drawn from a jar containing five red, four white, and three blue marbles.
Find the following probabilities using combinations.

9) P(all three red)	10) P(two white and 1 blue)
11) P(none white)	12) P(at least one red)

SECTION 8.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors.
Find the probabilities for the following events.

13) At least three freshmen.	14) No sophomores.
15) All four of the same class.	16) Not all four from the same class.
17) Exactly three of the same class.	18) More juniors than freshmen and sophomores combined.

Five cards are drawn from a deck. Find the probabilities for the following events.

19) Two hearts, two spades, and one club.	20) A flush of any suit (*all cards of a single suit*).
21) A full house of nines and tens (*3 nines and 2 tens*).	22) Any full house.
23) A pair of nines and a pair of tens (*and the fifth card is not a nine or ten*).	24) Any two pairs (*two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair*).

SECTION 8.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

Find the following probabilities:

25) P(playlist has 2 songs of each type)	26) P(playlist has no country songs)
27) P(playlist has 3 rock, 2 rap, and 1 country song)	28) P(playlist has 3 or 4 rock songs and the rest are rap songs)

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives. A committee of 5 people is selected to make a presentation to senior management.

Find the probabilities of the following events.

29) The committee has 2 engineers, 2 salespeople, and 1 customer service representative.	30) The committee contains 3 engineer and 2 salespeople.
31) The committee has no engineers.	32) The committee has all salespeople.

Do the following birthday problems.

33) If there are 5 people in a room, what is the probability that no two have the same birthday?	34) If there are 5 people in a room, find the probability that at least 2 have the same birthday.

8.4 Conditional Probability

In this section, you will learn to:

1. *recognize situations involving conditional probability*
2. *calculate conditional probabilities*

Suppose a friend asks you the probability that it will snow today.

If you are in Boston, Massachusetts in the winter, the probability of snow today might be quite substantial. If you are in Cupertino, California in summer, the probability of snow today is very tiny, this probability is pretty much 0.

Let A = the event that it will snow today

B = the event that today you are in Boston in wintertime

C = the event that today you are in Cupertino in summertime

Because the probability of snow is affected by the location and time of year, we can't just write P(A) for the probability of snow. We need to indicate the other information we know – location and time of year. We need to use **conditional probability**.

The event we are interested in is event A for snow. The other event is called the condition, representing location and time of year in this case.

We represent conditional probability using a vertical line | that means "if", or "given that", or "if we know that". The event of interest appears on the left of the |. The condition appears on the right side of the |.

The probability it will snow given that (if) you are in Boston in the winter is represented by **P(A|B).** In this case, the condition is B.

The probability that it will snow given that (if) you are in Cupertino in the summer is represented by **P(A|C)**. In this case, the condition is C.

Now, let's examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you $5, otherwise, you pay him $1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13.
This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only $1 to $5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of $1 to $5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol P(E | F) denotes the problem of finding the probability of E given that F has occurred. We read P(E | F) as "the probability of E, given F."

◆***Example 1*** A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

Solution: Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, P(E | F} = 2/4 or 1/2.

◆***Example 2*** One six sided die is rolled once.

a. Find the probability that the result is even.

b. Find the probability that the result is even given that the result is greater than three.

Solution: The sample space is S = {1,2,3,4,5,6}

Let event E be that the result is even and T be that the result is greater than 3.

a. P(E) = 3/6 because E = {2,4,6}

b. Because T = {4,5,6}, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, P(E|T} = 2/3

◆***Example 3*** A fair coin is tossed twice.

a. Find the probability that the result is is two heads.

b. Find the probability that the result is two heads given that at least one head is obtained.

Solution: The sample space is S = {HH, HT, TH, TT}

Let event E be that the two heads are obtained and F be at least one head is obtained

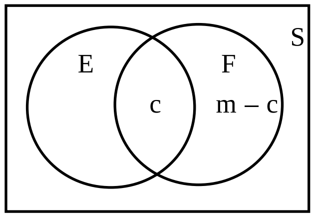
a. P(E) = 1/4 because E = {HH} and the sample space S has 4 outcomes.

b. F = {HH, HT, TH}. Since at least one head was obtained, TT did not occur. We are interested in the probability event E={HH} out of the 3 outcomes in the reduced sample space F.

Therefore, P(E|F} = 1/3

Let us now develop a formula for the conditional probability P(E | F).

Suppose an experiment consists of n equally likely events. Further suppose that there are m elements in F, and c elements in $E \cap F$, as shown in the following Venn diagram.



If the event F has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset F. Therefore, we only look at the set F and at nothing outside of F. Since F has m elements, the denominator in the calculation of P(E | F) is m. We may think that the numerator for our conditional probability is the number of elements in E. But clearly we cannot consider the elements of E that are not in F. We can only count the elements of E that are in F, that is, the elements in $E \cap F$. Therefore,

$$P(E \mid F) = \frac{c}{m}$$

Dividing both the numerator and the denominator by n, we get

$$P(E \mid F) = \frac{c/n}{m/n}$$

But $c/n = P(E \cap F)$, and $m/n = P(F)$.

Substituting, we derive the following formula for P(E | F).

Conditional Probability Rule

For Two Events E and F, the probability of "E Given F" is

$$\mathbf{P(E \mid F) = \frac{P(E \cap F)}{P(F)}}$$

◆***Example 4*** A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

Solution: Let E be the event that an even number shows, and F be the event that a number greater than three shows. We want P(E | F).

E = {2, 4, 6} and F = {4, 5, 6}. Which implies, $E \cap F = \{4, 6\}$

Therefore, P(F) = 3/6, and $P(E \cap F) = 2/6$

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

◆***Example 5*** The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

	Male(M)	Female(F)	Total
Public Transportation(T)	8	13	21
Drive(D)	39	40	79
Total	47	53	100

The events M, F, T, and D are self explanatory. Find the following probabilities.

a. P(D | M) b. P(F | D) c. P(M | T)

Solution: ***Solution 1:*** Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event M; we can look at only the "Male" column of the table and ignore the rest of the table: $P(D \mid M) = \frac{39}{47}$.

b. The condition is event D; we can look at only the "Drive" row of the table and ignore the rest of the table: $P(F \mid D) = \frac{40}{79}$.

c. The condition is event T; we can look at only the "Public Transportation" row of the table and ignore the rest of the table: $P(M \mid T) = \frac{8}{21}$.

Solution 2: We use the conditional probability formula $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$.

a. $P(D \mid M) = \frac{P(D \cap M)}{P(M)} = \frac{39/100}{47/100} = \frac{39}{47}$.

b. $P(F \mid D) = \frac{P(F \cap D)}{P(D)} = \frac{40/100}{79/100} = \frac{40}{79}$.

c. $P(M \mid T) = \frac{P(M \cap T)}{P(T)} = \frac{8/100}{21/100} = \frac{8}{21}$.

◆***Example 6*** Given P(E) = .5, P(F) = .7, and $P(E \cap F) = .3$. Find the following:

a. P(E | F) b. P(F | E).

Solution: We use the conditional probability formula.

a. $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{.3}{.7} = \frac{3}{7}$ b. $P(F \mid E) = \frac{P(E \cap F)}{P(E)} = .3/.5 = 3/5$

◆***Example 7*** E and F are mutually exclusive events such that P(E) = .4, P(F) = .9. Find P(E | F).

Solution: E and F are mutually exclusive, so $P(E \cap F) = 0$.

Therefore $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{0}{.9} = 0$.

◆***Example 8*** Given $P(F \mid E) = .5$, and $P(E \cap F) = .3$. Find P(E).

Solution: Using the conditional probability formula $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$, we get

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)}$$

Substituting and solving:

$$.5 = \frac{.3}{P(E)} \qquad \text{or} \qquad P(E) = 3/5$$

◆***Example 9*** In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

Solution: Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want $P(E \mid F)$.

We list the sample space along with the events E and F.

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

E = {BBG, BGB, GBB} and F = {BBB , BBG, BGB, GBB}

$E \cap F$ = {BBG, BGB, GBB}

Therefore, $P(F) = 4/8$, and $P(E \cap F) = 3/8$, and

$$P(E \mid F) = \frac{P(E \cap F)}{P(E)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

◆***Example 10*** At a community college 65% of the students subscribe to Amazon Prime, 50% subscribe to Netflix, and 20% subscribe to both.
If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that he subscribes to Netflix

b. the student subscribes to Netflix given that he subscribes to Amazon Prime

Solution: Let A be the event that the student subscribes to Amazon Prime, and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = P(A) = 0.65

P(student subscribes to Netflix) = P(N) = 0.50

P(student subscribes to both Amazon Prime and Netflix) = $P(A \cap N) = 0.20$

Then use the conditional probability rule:

a. $P(A \mid N) = \frac{P(A \cap N)}{P(N)} = \frac{.20}{.50} = \frac{2}{5}$

b. $P(N \mid A) = \frac{P(A \cap N)}{P(A)} = \frac{.20}{.65} = \frac{4}{13}.$

SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY

Questions 1 – 4: Do these problems using the conditional probability formula: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$.

1) A card is drawn from a deck. Find the conditional probability of P(a queen | a face card).	2) A card is drawn from a deck. Find the conditional probability of P(a queen | a club).
3) A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown.	4) If P(A) = .3 , P(B) = .4, P(A and B) = .12, find: a) P(A | B) b) P(B | A)

Questions 5 – 8 refer to the following: The table shows the distribution of Democratic and Republican U.S. Senators by gender in the 114^{th} Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities:

5) P(M | D)	6) P(D | M)
7) P(F | R)	8) P(R | F)

SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY

Do the following conditional probability problems.

9) At a college, 20% of the students take Finite Math, 30% take History, and 5% take both Finite Math and History. If a student is chosen at random, find the following conditional probabilities. a) He is taking Finite Math given that he is taking History. b) He is taking History assuming that he is taking Finite Math.	10) At a college, 60% of the students pass Accounting, 70% pass English, and 30% pass both of these courses. If a student is selected at random, find the following conditional probabilities. a) He passes Accounting given that he passed English. b) He passes English assuming that he passed Accounting.
11) If P(F) = .4 , P(E | F) = .3, find P(E and F).	12) P(E) = .3, P(F) = .3; E and F are mutually exclusive. Find P(E | F).
13) If P(E) = .6 , P(E and F) = .24, find P(F | E).	14) If P(E and F) = .04 , P(E | F) = .1, find P(F).

At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper.

15) Find the probability that a course has a final exam given that it has a research paper.	16) Find the probability that a course has a research paper if it has a final exam.

SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY

Consider a family of three children. Find the following probabilities.

17) P(two boys | first born is a boy)	18) P(all girls | at least one girl is born)
19) P(children of both sexes | first born is a boy)	20) P(all boys | there are children of both sexes)

Questions 21 – 27 refer to the following:
The table shows highest attained educational status for a sample of US residents age 25 or over:
based on data from http://www.census.gov/hhes/socdemo/education/data/cps/2010/Table1-01.xls

	(D) Did not Complete High School	(H) High School Graduate	(C) Some College	(A) Associate Degree	(B) Bachelor Degree	(G) Graduate Degree	TOTAL
25-44 (R)	95	228	143	81	188	61	796
45-64 (S)	83	256	136	80	150	67	772
65+ (T)	96	191	84	36	80	41	528
Total	274	675	363	197	418	169	2096

Use this table to determine the following probabilities:

21) P(C | T)	22) P(S | A)	23) P(C and T)
24) P(R | B)	25) P(B | R)	26) P(G|S)

8.5 Independent Events

In this section, you will:

1. *define independent events*
2. *identify whether two events are independent or dependent*

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example 1 we revisit the discussion at the beginning of the previous section and then contrast that with Example 2.

◆***Example 1*** A card is drawn from a deck. Find the following probabilities.

a. The card is a king. b. The card is a king given that the card is a face card.

Solution: a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

P(The card is a king | The card is a face card) = 4/12 = 1/3.

The reader should observe that in the above example,

P(The card is a king | The card is a face card) ≠ P(The card is a king)

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

◆***Example 2*** A card is drawn from a deck. Find the following probabilities.

a. The card is a king. b. The card is a king given that a red card has shown.

Solution: a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

P(The card is a king | A red card has shown) = 2/26 = 1/13.

The reader should observe that in the above example,

P(The card is a king | A red card has shown) = P(The card is a king)

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

> Two Events E and F are **independent** if and only if at least one of the following two conditions is true.
>
> **1. P(E | F) = P(E) or 2. P(F | E) = P(F)**
>
> If the events are not independent, then they are dependent.
>
> If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Multiplying both sides by P(F), we get

$$P(E \cap F) = P(E \mid F)\, P(F)$$

Now if the two events are independent, then by definition

$$P(E \mid F) = P(E)$$

Substituting, $P(E \cap F) = P(E)\,P(F)$

We state it formally as follows.

> **Test For Independence**
>
> Two events E and F are independent if and only if
>
> **P(E ∩ F) = P(E) P(F)**

In the Examples 3 and 4, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F)=P(E)$ or $P(F|E)=P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

◆***Example 3*** The table below shows the distribution of color-blind people by gender.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $P(C \cap M) = P(C)P(M)$.

From the table: $P(C) = 7/100$, $P(M) = 52/100$ and $P(C \cap M) = 6/100$

So $P(C)\,P(M) = (7/100)(52/100) = .0364$

which is **<u>not</u>** equal to $P(C \cap M) = 6/100 = .06$

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C) = 7/100 = 0.07$

From the male column $P(C|M) = 6/52 = 0.1154$

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

◆***Example 4*** In a city with two airports, 100 flights were surveyed. 20 of those flights departed late. 45 flights in the survey departed from airport A; 9 of those flights departed late. 55 flights in the survey departed from airport B; 11 flights departed late. Are the events "depart from airport A" and "departed late" independent?

Solution 1: Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$$P(A \cap L) = 9/100, \quad P(A) = 45/100 \text{ and } P(L) = 20/100$$

In order for two events to be independent, we must have $P(A \cap L) = P(A)\,P(L)$

Since $P(A \cap L) = 9/100 = 0.09$

and $P(A)\,P(L) = (45/100)(20/100) = 900/10000 = 0.09$

the two events "departing from airport A" and "departing late" are independent.

Solution 2: The definition of independent events states that two events are independent if $P(E|F)=P(E)$.

In this problem we are given that

$$P(L|A) = 9/45 = 0.2 \quad \text{and} \quad P(L) = 20/100 = 0.2$$

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

◆***Example 5*** A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F b. F and G c. E and G

Solution: We list the sample space, the events, their intersections and the probabilities.

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

E = {HHH, HHT, HTH, HTT}, $P(E) = 4/8$ or $1/2$

F = {HHH, HHT, HTH, THH}, $P(F) = 4/8$ or $1/2$

G = {HHT, THH}, $P(G) = 2/8$ or $1/4$

$E \cap F$= {HHH, HHT, HTH}, $P(E \cap F) = 3/8$

$F \cap G$ = {HHT, THH}, $P(F \cap G) = 2/8$ or $1/4$

$E \cap G$ = {HHT} $P(E \cap G) = 1/8$

a. E and F will be independent if and only if $P(E \cap F) = P(E)\,P(F)$

$P(E \cap F) = 3/8$ and $P(E)\,P(F) = 1/2 \cdot 1/2 = 1/4$.

Since $3/8 \neq 1/4$, we have $P(E \cap F) \neq P(E)\,P(F)$.

Events E and F are not independent.

b. F and G will be independent if and only if $P(F \cap G) = P(F)\,P(G)$.

$P(F \cap G) = 1/4$ and $P(F)\,P(G) = 1/2 \cdot 1/4 = 1/8$

Since $3/8 \neq 1/4$, we have $P(F \cap G) \neq P(F)\,P(G)$.

Events F and G are not independent.

c. E and G will be independent if $P(E \cap G) = P(E)\,P(G)$

$P(E \cap G) = 1/8$ and $P(E)\,P(G) = 1/2 \cdot 1/4 = 1/8$

Events E and G are independent events because $P(E \cap G) = P(E)\,P(G)$

◆***Example 6*** The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

Solution: Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given $P(A) = .30$ and $P(R) = .50$, and we want to find $P(A \cap R)$.

Since we are told that the events A and R are independent,

$$P(A \cap R) = P(A)\,P(R) = (.30)(.50) = .15$$

◆***Example 7*** Given $P(B \mid A) = .4$. If A and B are independent, find P(B).

Solution: If A and B are independent, then by definition $P(B \mid A) = P(B)$

Therefore, $P(B) = .4$

◆***Example 8*** Given $P(A) = .7$, $P(B \mid A) = .5$. Find $P(A \cap B)$.

Solution 1: By definition $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

Substituting, we have

$$.5 = \frac{P(A \cap B)}{.7}$$

Therefore, $P(A \cap B) = .35$

Solution 2: Again, start with $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

Multiplying both sides by P(A) gives

$$P(A \cap B) = P(B \mid A)\, P(A) = (.5)(.7) = .35$$

Both solutions to Example 8 are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A \cap B)$. That gives the following result:

Multiplication Rule for events that are NOT independent
If events E and F are not independent
$\mathbf{P(E \cap F) = P(E \mid F)\, P(F)}$ **and** $\mathbf{P(E \cap F) = P(F \mid E)\, P(E)}$

◆***Example 9*** Given $P(A) = .5$, $P(A \cup B) = .7$, if A and B are independent, find P(B).

Solution: The addition rule states that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since A and B are independent, $P(A \cap B) = P(A)\, P(B)$

We substitute for $P(A \cap B)$ in the addition formula and get

$$P(A \cup B) = P(A) + P(B) - P(A)\, P(B)$$

By letting $P(B) = x$, and substituting values, we get

$$.7 = .5 + x - .5x$$

$$.7 = .5 + .5x$$

$$.2 = .5x$$

$$.4 = x$$

Therefore, $P(B) = .4$

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

	MAIN (M)	BRANCH (B)	TOTAL
FICTION (F)	300	100	400
NON-FICTION (N)	150	50	200
TOTALS	450	150	600

Use this table to determine the following probabilities:

1) P(F)	2) P(M | F)
3) P(N | B)	4) Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion.

For a two-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

5) Find the following. a) P(E) b) P(F) c) P(E ∩ F) d) Are E and F independent? Use probabilities to justify your conclusion.	6) Find the following. a) P(F) b) P(G) c) P(F ∩ G) d) Are F and G independent? Use probabilities to justify your conclusion.

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

Do the following problems involving independence.

7) If P(E) = .6, P(F) = .2, and E and F are independent, find P(E and F).	8) If P(E) = .6, P(F) = .2, and E and F are independent, find P(E or F).
9) If P(E) = .9, P(F | E) = .36, and E and F are independent, find P(F).	10) If P(E) = .6, P(E or F) = .8, and E and F are independent, find P(F).
11) In a survey of 100 people, 40 were casual drinkers, and 60 did not drink. Of the ones who drank, 6 had minor headaches. Of the non-drinkers, 9 had minor headaches. Are the events "drinkers" and "had headaches" independent?	12) It is known that 80% of the people wear seat belts, and 5% of the people quit smoking last year. If 4% of the people who wear seat belts quit smoking, are the events, wearing a seat belt and quitting smoking, independent?

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

13) John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities.
    a) P(both of them will pass statistics)
    b) P(at least one of them will pass statistics)

14) Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities:
    a) P(Jane will make both connections)
    b) P(Jane will make at least one connection)

For a three-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

15) Find the following.
    a) P(E)
    b) P(F)
    c) $P(E \cap F)$
    d) Are E and F independent?

16) Find the following.
    a) P(F)
    b) P(G)
    c) $P(F \cap G)$
    d) Are F and G independent?

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

17) P(K|D) = 0.7, P(D) = 0.25 and P(K)=0.7

a. Are events K and D independent?
Use probabilities to justify your conclusion.

b. Find P(K ∩D)

18) P(R|S) = 0.4, P(S) = 0.2 and P(R)=0.3

a. Are events R and S independent?
Use probabilities to justify your conclusion.

b. Find P(R ∩S)

19) At a college:
54% of students are female
25% of students are majoring in engineering.
15% of female students are majoring in engineering.

Event E = student is majoring in engineering
Event F = student is female

a. Are events E and F independent?
Use probabilities to justify your conclusion.

b. Find P(E ∩F)

20) At a college:
54% of all students are female
60% of all students receive financial aid.
60% of female students receive financial aid.

Event A = student receives financial aid
Event F = student is female

a. Are events A and F independent?
Use probabilities to justify your conclusion.

b. Find P(A ∩F)

SECTION 8.6 PROBLEM SET: CHAPTER REVIEW

1) Two dice are rolled. Find the probability that the sum of the dice is

a) four
b) five

2) A jar contains 3 red, 4 white, and 5 blue marbles.
If a marble is chosen at random, find the following probabilities:

a) P(red or blue)
b) P(not blue)

3) A card is drawn from a standard deck. Find the following probabilities:

a) P(a jack or a king)
b) P(a jack or a spade)

4) A basket contains 3 red and 2 yellow apples. Two apples are chosen at random.
Find the following probabilities:

a) P(one red, one yellow)
b) P(at least one red)

5) A basket contains 4 red, 3 white, and 3 blue marbles. Three marbles are chosen at random.
Find the following probabilities:

a) P(two red, one white)
b) P(first red, second white, third blue)
c) P(at least one red)
d) P(none red)

6) Given a family of four children. Find the following probabilities:

a) P(All boys)
b) P(1 boy and 3 girls)

7) Consider a family of three children. Find the following:

a) P(children of both sexes | first born is a boy)
b) P(all girls | children of both sexes)

8) Mrs. Rossetti is flying from San Francisco to New York. On her way to the San Francisco Airport she encounters heavy traffic and determines that there is a 20% chance that she will be late to the airport and will miss her flight. Even if she makes her flight, there is a 10% chance that she will miss her connecting flight at Chicago. What is the probability that she will make it to New York as scheduled?

9) At a college, twenty percent of the students take history, thirty percent take math, and ten percent take both. What percent of the students take at least one of these two courses?

10) In a T-maze, a mouse may run to the right (R) or may run to the left (L). A mouse goes up the maze three times, and events E and F are described as follows:

E: Runs to the right on the first trial
F: Runs to the left two consecutive times

Determine whether the events E and F are independent.

11) A college has found that 20% of its students take advanced math courses, 40% take advanced English courses and 15% take both advanced math and advanced English courses. If a student is selected at random, what is the probability that

a) he is taking English given that he is taking math?
b) he is taking math or English?

12) If there are 35 students in a class, what is the probability that at least two have the same birthday?

13) A student feels that her probability of passing accounting is .62, of passing mathematics is .45, and her passing accounting or mathematics is .85. Find the probability that the student passes both accounting and math.

SECTION 8.6 PROBLEM SET: CHAPTER REVIEW

14) There are nine judges on the U. S. Supreme Court. Suppose that five are conservative and four are liberal. This year the court will act on six major cases. What is the probability that out of six cases the court will favor the conservatives in at least four?

15) Five cards are drawn from a deck. Find the probability of obtaining

a) four cards of a single suit

b) two cards of one suit, two of another suit, and one from the remaining

c) a pair(e.g. two aces and three other cards)

d) a straight flush(five in a row of a single suit but not a royal flush)

16) The following table shows a distribution of drink preferences by gender.

	Coke(C)	Pepsi(P)	Seven Up(S)	TOTALS
Male(M)	60	50	22	132
Female(F)	50	40	18	108
TOTALS	110	90	40	240

The events M, F, C, P and S are defined as Male, Female, Coca Cola, Pepsi, and Seven Up, respectively. Find the following:

a) P(F | S)

b) P(P | F)

c) P(C | M)

d) P(M | P U C)

e) Are the events F and S mutually exclusive?

f) Are the events F and S independent?

17) At a clothing outlet 20% of the clothes are irregular, 10% have at least a button missing and 4% are both irregular and have a button missing. If Martha found a dress that has a button missing, what is the probability that it is irregular?

18) A trade delegation consists of four Americans, three Japanese and two Germans. Three people are chosen at random. Find the following probabilities:

a) P(two Americans and one Japanese)

b) P(at least one American)

c) P(One of each nationality)

d) P(no German)

19) A coin is tossed three times, and the events E and F are as follows.

E: It shows a head on the first toss

F: Never turns up a tail

Are the events E and F independent?

20) If P(E) = .6 and P(F) = .4 and E and F are mutually exclusive, find P(E and F).

21) If P(E)=.5 and P(F)=.3 and E and F are independent, find P(E U F).

22) If P(F)=.9 and P(E | F)=.36 and E and F are independent, find P(E).

23) If P(E)=.4 and P(E or F) =.9 and E and F are independent, find P(F).

24) If P(E) = .4 and P(F | E) = .5, find P(E and F).

25) If P(E) = .6 and P(E and F) = .3, find P(F | E).

26) If P(E) = .3 and P(F) = .4 and E and F are independent, find P(E | F).

CHAPTER 9

Chapter 9: More Topics in Probability

In this chapter, you will learn to:

1. *Find the probability of a binomial experiment.*
2. *Find probabilities using Bayes' Formula.*
3. *Find the expected value or payoff in a game of chance.*
4. *Find probabilities using tree diagrams.*

9.1 Binomial Probability

In this section you will learn to:

1. *recognize when to use the binomial probability distribution*
2. *derive the formula for the binomial probability distribution*
3. *calculate probabilities for a binomial probability experiment*

In this section, we consider problems that involve a sequence of trials, where each trial has only two outcomes, a *success* or a *failure*. These trials are independent, that is, the outcome of one does not affect the outcome of any other trial. The probability of success, p, and the probability of failure, (1 – p), remains the same throughout the experiment. These problems are called **binomial probability** problems. Since these problems were researched by Swiss mathematician Jacques Bernoulli around 1700, they are also called **Bernoulli trials.**

We give the following definition:

Binomial Experiment

A binomial experiment satisfies the following four conditions:

1. There are only two outcomes, a success or a failure, for each trial.
2. The same experiment is repeated several times.
3. The trials are independent; that is, the outcome of a particular trial does not affect the outcome of any other trial.
4. The probability of success remains the same for every trial.

This probability model that will give us the tools to solve many real-life problems , such as:

1. If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?
2. If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 7 out of 10 free throws in a game?
3. If a medicine cures 80% of the people who take it, what is the probability that among the ten people who take the medicine, 6 will be cured?
4. If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?
5. If a telemarketing executive has determined that 15% of the people contacted will purchase the product, what is the probability that among the 12 people who are contacted, 2 will buy the product?

We now consider the following example to develop a formula for finding the probability of k successes in n Bernoulli trials.

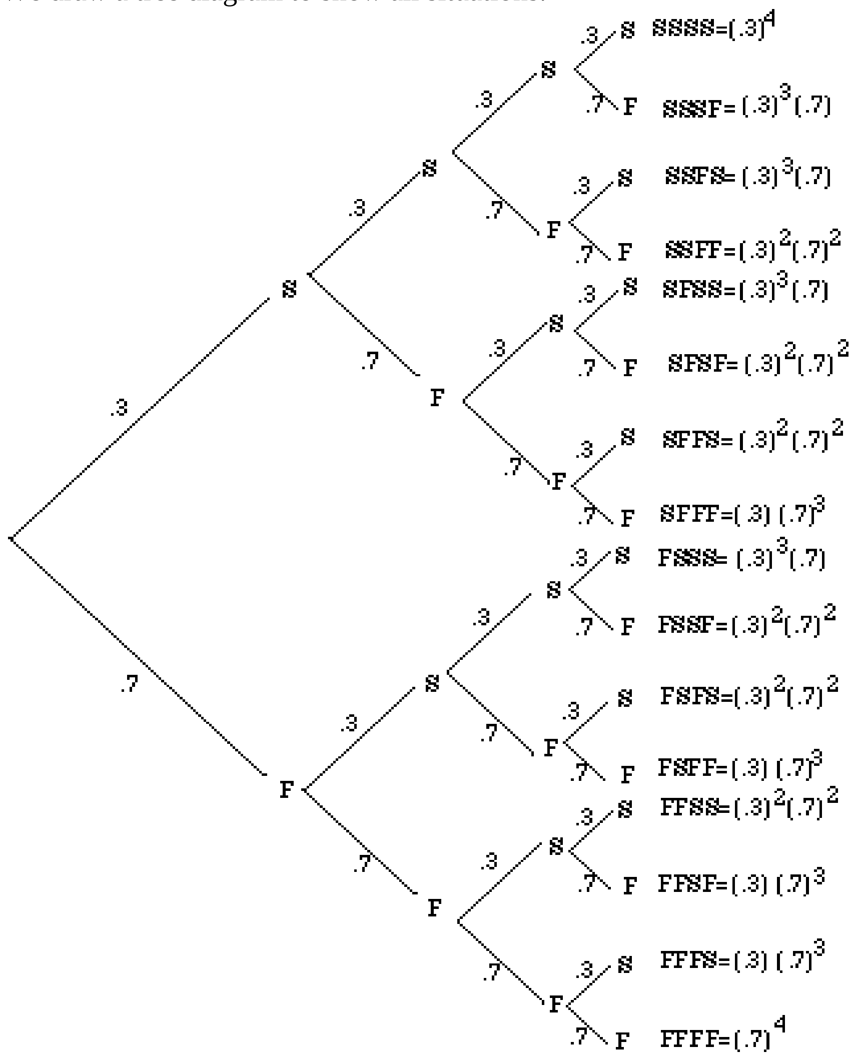
◆***Example 1*** A baseball player has a batting average of .300. If he bats four times in a game, find the probability that he will have

a. 4 hits b. 3 hits c. 2 hits d. 1 hit e. no hits.

Solution: Let S denote that the player gets a hit, and F denote that he does not get a hit.

This is a binomial experiment because it meets all four conditions. First, there are only two outcomes, S or F. Clearly the experiment is repeated four times. Lastly, if we assume that the player's skillfulness to get a hit does not change each time he comes to bat, the trials are independent with a probability of .3 of getting a hit during each trial.

We draw a tree diagram to show all situations.



Let us first find the probability of getting, for example, two hits. We will have to consider the six possibilities, SSFF, SFSF, SFFS, FSSF, FSFS, FFSS, as shown in the above tree diagram. We list the probabilities of each below.

$P(SSFF) = (.3)(.3)(.7)(.7) = (.3)^2(.7)^2$

$P(SFSF) = (.3)(.7)(.3)(.7) = (.3)^2(.7)^2$

$P(SFFS) = (.3)(.7)(.7)(.3) = (.3)^2(.7)^2$

$P(FSSF) = (.7)(.3)(.3)(.7) = (.3)^2(.7)^2$

$P(FSFS) = (.7)(.3)(.7)(.3) = (.3)^2(.7)^2$

$P(FFSS) = (.7)(.7)(.3)(.3) = (.3)^2(.7)^2$

Since the probability of each of these six outcomes is $(.3)^2(.7)^2$, the probability of obtaining two successes is $6(.3)^2(.7)^2$.

The probability of getting one hit can be obtained in the same way. Since each permutation has one S and three F's, there are four such outcomes: SFFF, FSFF, FFSF, and FFFS.

And since the probability of each of the four outcomes is $(.3)(.7)^3$, the probability of getting one hit is $4(.3)(.7)^3$.

The table below lists the probabilities for all cases, and shows a comparison with the binomial expansion of fourth degree. Again, p denotes the probability of success, and q = (1 – p) the probability of failure.

Outcome	Four Hits	Three Hits	Two Hits	One Hit	No Hits
Probability	$(.3)^4$	$4\ (.3)^3(.7)$	$6\ (.3)^2(.7)^2$	$4\ (.3)(.7)^3$	$(.7)^4$

$$(.3 + .7)^4 = (.3)^4 + 4(.3)^3(.7) + 6(.3)^2(.7)^2 + 4\,(.3)(.7)^3 + (.7)^4$$

$$(p + q)^4 = p^4 + 4\,p^3q + 6\,P^2q^2 + 4\,pq^3 + q^4$$

This gives us the following theorem:

Binomial Probability Theorem

The probability of obtaining k successes in n independent Bernoulli trials is given by

$$\mathbf{P(n, k; p) = nCk\ p^k q^{n-k}}$$

where p denotes the probability of success and q = (1 – p) the probability of failure.

We use the binomial probability formula to solve the following examples.

◆***Example 2*** If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?

Solution: Let S denote the probability of obtaining a head, and F the probability of getting a tail.

Clearly, n = 10, k = 3, p = 1/2, and q = 1/2.

Therefore, $b(10, 3; 1/2) = 10C3\,(1/2)^3(1/2)^7 = .1172$

◆***Example 3*** If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 6 out of 10 free throws in a game?

Solution: The probability of making a free throw is 3/4.
Therefore, p = 3/4, q = 1/4, n = 10, and k = 6.

Therefore, $b(10, 6; 3/4) = 10C6\,(3/4)^6(1/4)^4 = .1460$

◆***Example 4*** If a medicine cures 80% of the people who take it, what is the probability that of the eight people who take the medicine, 5 will be cured?

Solution: Here p =.80, q = .20, n = 8, and k = 5.

$$b(8, 5; .80) = 8C5\,(.80)^5(.20)^3 = .1468$$

◆***Example 5*** If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?

Solution: If S denotes the probability that the chip is defective, and F the probability that the chip is not defective, then p = .04, q = .96, n = 60, and k = 3.

$$b(60, 3; .04) = 60C3\,(.04)^3(.96)^{57} = .2138$$

◆***Example 6*** A telemarketing executive has determined that 15% of people contacted will purchase the product. 12 people are contacted about this product.
(a) Find the probability that among 12 people contacted, 2 will buy the product.
(b) Find the probability that among 12 people contacted, at most 2 will buy the product?

Solution: (a) If S denoted the probability that a person will buy the product, and F the probability that the person will not buy the product, then p = .15, q = .85, n = 12, and k = 2.

$$b(12, 2, .15) = 12C2\,(.15)^2(.85)^{10} = .2924.$$

The probability that 2 people buy the product is 0.2924.

(b) Again p = .15, q = .85, n = 12. But to find the probability that **at most 2** buy the product, we need to find the probabilities for k= 0, k=1, k=2 and add them together.

$$b(12, 0, .15) = 12C0\,(.15)^0(.85)^{12} = .1422$$

$$b(12, 1, .15) = 12C1\,(.15)^1(.85)^{11} = .3012$$

Adding all three probabilities gives: .1422 +0.3012 +.2924 = .7358

The probability that at most 2 people buy the product is 0.7358.

SECTION 9.1 PROBLEM SET: BINOMIAL PROBABILITY

Do the following problems using the binomial probability formula.

1) A coin is tossed ten times. Find the probability of getting six heads and four tails.	2) A family has three children. Find the probability of having one boy and two girls.
3) What is the probability of getting three aces(ones) if a die is rolled five times?	4) A baseball player has a .250 batting average. What is the probability that he will have three hits in five times at bat?
5) A basketball player has an 80% chance of sinking a basket on a free throw. What is the probability that he will sink at least three baskets in five free throws?	6) With a new flu vaccination, 85% of the people in the high risk group can go through the entire winter without contracting the flu. In a group of six people who were vaccinated with this drug, what is the probability that at least four will not get the flu?

SECTION 9.1 PROBLEM SET: BINOMIAL PROBABILITY

Do the following problems using the binomial probability formula.

7) A transistor manufacturer has known that 5% of the transistors produced are defective. What is the probability that a batch of twenty five transistors will have two defective?	8) It has been determined that only 80% of the people wear seat belts. If a police officer stops a car with four people, what is the probability that at least one person will not be wearing a seat belt?
9) What is the probability that a family of five children will have at least three boys?	10) What is the probability that a toss of four coins will yield at most two heads?
11) A telemarketing executive has determined that for a particular product, 20% of the people contacted will purchase the product. If 10 people are contacted, what is the probability that at most 2 will buy the product?	12) To the problem: "Five cards are dealt from a deck of cards, find the probability that three of them are kings," the following incorrect answer was offered by a student. $5C3\ (1/13)^3(12/13)^2$ What change would you make in the wording of the problem for the given answer to be correct?

SECTION 9.1 PROBLEM SET: BINOMIAL PROBABILITY

Do the following problems using the binomial probability formula.

13) 63% of all registered voters in a large city voted in the last election. 20 registered voters from this city are randomly selected. Find the probability that a. exactly half of them voted in the last election. b. all of them voted	14) 30% of customers at BigMart pay cash for their purchases. Suppose that 15 customers are randomly selected. Find the probability that a. 5 or 6 of them pay cash b. at most 1 pays cash
15) 12% of all cars on Brighton Expressway exceed the speed limit. If 10 vehicles on this road are randomly selected and their speed is recorded by radar, find the probability that a. none of them are exceeding the speed limit b. 1 or 2 are exceeding the speed limit.	16) Suppose that 73% of all people taking a professional certification exam pass the exam. If 12 people who take this exam are randomly selected, find the probability that a. exactly half of them pass the exam b. all of them pass the exam c. 8 or 9 of them pass the exam

9.2 Bayes' Formula

In this section you will learn to:

1. *find probabilities using Bayes' formula*
2. *use a probability tree to find and represent values needed when using Bayes' formula.*

In this section, we will develop and use Bayes' Formula to solve an important type of probability problem. Bayes' formula is a method of calculating the conditional probability P(F | E) from P(E | F). The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' formula does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

◆***Example 1*** Suppose you are given two jars. Jar I contains one black and 4 white marbles, and Jar II contains 4 black and 6 white marbles. If a jar is selected at random and a marble is chosen,

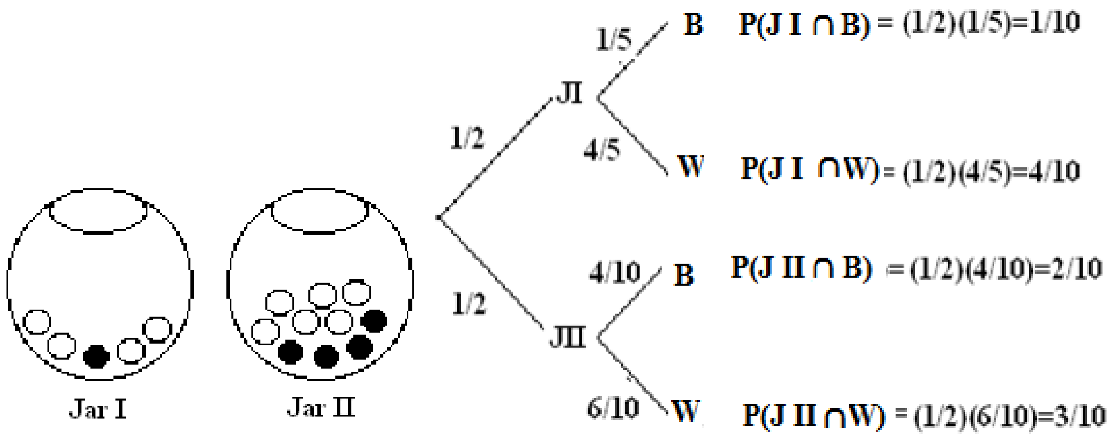
a. What is the probability that the marble chosen is a black marble?

b. If the chosen marble is black, what is the probability that it came from Jar I?

c. If the chosen marble is black, what is the probability that it came from Jar II?

Solution: Let J I be the event that Jar I is chosen, J II be the event that Jar II is chosen, B be the event that a black marble is chosen and W the event that a white marble is chosen.

We illustrate using a tree diagram.



a. The probability that a black marble is chosen is P(B) = 1/10 + 2/10 = 3/10.

b. To find P(J I | B), we use the definition of conditional probability, and we get

$$P(J\ I \mid B) = \frac{P(J\ I \cap B)}{P(B)} = \frac{1/10}{3/10} = \frac{1}{3}$$

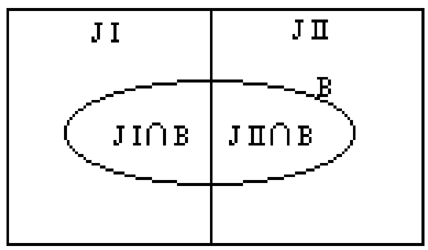
c. Similarly, $P(J\ II \mid B) = \frac{P(J\ II \cap B)}{P(B)} = \frac{2/10}{3/10} = \frac{2}{3}$

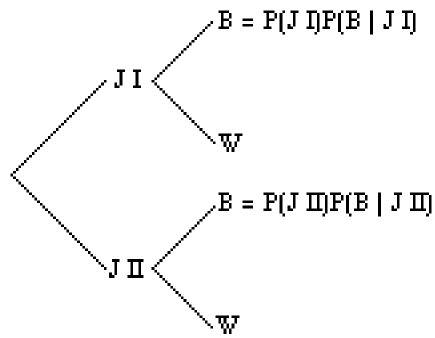
In parts b and c, the reader should note that the denominator is the sum of all probabilities of all branches of the tree that produce a black marble, while the numerator is the branch that is associated with the particular jar in question.

We will soon discover that this is a statement of Bayes' formula .

Let us first visualize the problem.

We are given a sample space S and two mutually exclusive events J I and J II. That is, the two events, J I and J II, divide the sample space into two parts such that J I $\cup$ J II = S. Furthermore, we are given an event B that has elements in both J I and J II, as shown in the Venn diagram below.





From the Venn diagram, we can see that $B = (B \cap J\ I) \cup (B \cap J\ II)$

Therefore:

$$P(B) = P(B \cap J\ I) + P(B \cap J\ II) \qquad (1)$$

But the product rule in chapter 7 gives us

$$P(B \cap J\ I) = P(J\ I) \cdot P(B \mid J\ I) \quad \text{and} \quad P(B \cap J\ II) = P(J\ II) \cdot P(B \mid J\ II)$$

Substituting in (1), we get

$$P(B) = P(J\ I) \cdot P(B \mid J\ I) + P(J\ II) \cdot P(B \mid J\ II)$$

The conditional probability formula gives us

$$P(J\ I \mid B) = \frac{P(J\ I \cap B)}{P(B)}$$

Therefore, $P(J\ I \mid B) = \dfrac{P(J\ I) \cdot P(B \mid J\ I)}{P(B)}$

or, $P(J\ I \mid B) = \dfrac{P(J\ I) \cdot P(B \mid J\ I)}{P(J\ I) \cdot P(B \mid J\ I) + P(J\ II) \cdot P(B \mid J\ II)}$

The last statement is Bayes' Formula for the case where the sample space is divided into two partitions.

The following is the generalization of Bayes' formula for n partitions.

> Let S be a sample space that is divided into n partitions, $A_1, A_2, \ldots A_n$. If E is any event in S, then
>
> $$\mathbf{P(A_i \mid E) = \frac{P(A_i)\, P(E \mid A_i)}{P(A_1)\, P(E \mid A_1) + P(A_2)\, P(E \mid A_2) + \cdots + P(A_n)\, P(E \mid A_n)}}$$

We begin with the following example.

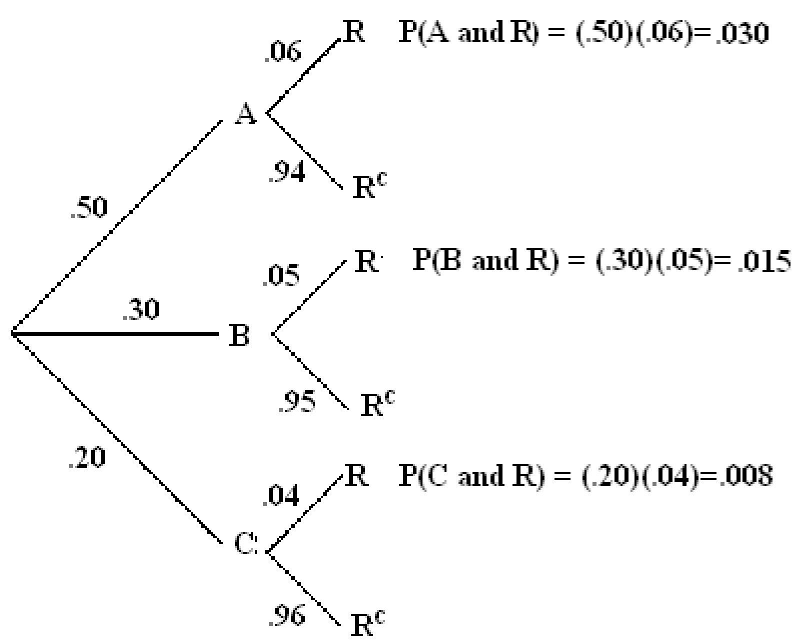
◆***Example 2*** A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

Solution: Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and by using Bayes' formula.

We draw a tree diagram.



The probability P(A | R), for example, is a fraction whose denominator is the sum of all probabilities of all branches of the tree that result in an appliance that needs repair before the warranty expires, and the numerator is the branch that is associated with Manufacturer A. P(B | R) and P(C | R) are found in the same way.

$$P(A \mid R) = \frac{.030}{(.030) + (.015) + (.008)} = \frac{.030}{.053} = .566$$

$$P(B \mid R) = \frac{.015}{.053} = .283 \text{ and } P(C \mid R) = \frac{.008}{.053} = .151.$$

Alternatively, using Bayes' formula,

$$P(A \mid R) = \frac{P(A)P(R \mid A)}{P(A)P(R \mid A) + P(B)P(R \mid B) + P(C)P(R \mid C)}$$

$$= \frac{.030}{(.030) + (.015) + (.008)} = \frac{.030}{.053} = .566$$

P(B | R) and P(C | R) can be determined in the same manner.

◆***Example 3*** There are five Jacy's department stores in San Jose. The distribution of number of employees by gender is given in the table below.

Store Number	Number of Employees	Percent of Women Employees
1	300	.40
2	150	.65
3	200	.60
4	250	.50
5	100	.70
	Total = 1000	

If an employee chosen at random is a woman, what is the probability that the employee works at store III?

Solution: Let k = 1, 2, . . . , 5 be the event that the employee worked at store k, and W be the event that the employee is a woman. Since there are a total of 1000 employees at the five stores,

P(1) = .30 P(2) = .15 P(3) = .20 P(4) = .25 P(5) = .10

Using Bayes' formula,

$$P(3 \mid W) = \frac{P(3)P(W \mid 3)}{P(1)P(W \mid 1) + P(2)P(W \mid 2) + P(3)P(W \mid 3) + P(4)P(W \mid 4) + P(5)P(W \mid 5)}$$

$$= \frac{(.20)(.60)}{(.30)(.40) + (.15)(.65) + (.20)(.60) + (.25)(.50) + (.10)(.70)}$$

$$= .2254$$

SECTION 9.2 PROBLEM SET: BAYES' FORMULA

1) Jar I contains five red and three white marbles, and Jar II contains four red and two white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram below, and find the following probabilities.

a) P(marble is red)

b) P(It came from Jar II | marble is white)

c) P(Red | Jar I)

2) In Mr. Symons' class, if a student does homework most days, the chance of passing the course is 90%. On the other hand, if a student does not do homework most days, the chance of passing the course is only 20%.

H = event that the student did homework

C = event that the student passed the course

Mr. Symons claims that 80% of his students do homework on a regular basis. If a student is chosen at random from Mr. Symons' class, find the following probabilities.

a) P(C)

b) P(H |C)

c) P(C|H)

3) A city has 60% Democrats, and 40% Republicans. In the last mayoral election, 60% of the Democrats voted for their Democratic candidate while 95% of the Republicans voted for their candidate. Which party's mayor runs city hall?

4) In a certain population of 48% men and 52% women, 56% of the men and 8% of the women are color-blind.

a) What percent of the people are color-blind?

b) If a person is found to be color-blind, what is the probability that the person is a male?

SECTION 9.2 PROBLEM SET: BAYES' FORMULA

5) A test for a certain disease gives a positive result 95% of the time if the person actually carries the disease. However, the test also gives a positive result 3% of the time when the individual is not carrying the disease. It is known that 10% of the population carries the disease. If a person tests positive, what is the probability that he or she has the disease?

6) A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair?

7) A computer company buys its chips from three different manufacturers. Manufacturer I provides 60% of the chips and is known to produce 5% defective; Manufacturer II supplies 30% of the chips and makes 4% defective; while the rest are supplied by Manufacturer III with 3% defective chips. If a chip is chosen at random, find the following probabilities:

a) P(the chip is defective)

b) P(chip is from Manufacturer II | defective)

c) P(defective |chip is from manufacturer III)

8) Lincoln Union High School District is made up of three high schools: Monterey, Fremont, and Kennedy, with an enrollment of 500, 300, and 200, respectively. On a given day, the percentage of students absent at Monterey High School is 6%, at Fremont 4%, and at Kennedy 5%. If a student is chosen at random, find the probabilities below:
Hint: Convert the enrollments into percentages.

a) P(the student is absent)

b) P(student is from Kennedy | student is absent)

c) P(student is absent | student is from Fremont)

SECTION 9.2 PROBLEM SET: BAYES' FORMULA

9) At a retail store, 20% of customers use the store's online app to assist them when shopping in the store ; 80% of store shoppers don't use the app.

Of those customers that use the online app while in the store, 50% are very satisfied with their purchases, 40% are moderately satisfied, and 10% are dissatisfied.

Of those customers that do not use the online app while in the store, 30% are very satisfied with their purchases, 50% are moderately satisfied and 20% are dissatisfied.

Indicate the events by the following:
A = shopper uses the app in the store
N = shopper does not use the app in the store
V = very satisfied with purchase
M = moderately satisfied
D = dissatisfied

a. Find P(A and D), the probability that a store customer uses the app and is dissatisfied

b. Find P(A|D), the probability that a store customer uses the app if the customer is dissatisfied.

10**)** A medical clinic uses a pregnancy test to confirm pregnancy in patients who suspect they are pregnant. Historically data has shown that overall, 70% of the women at this clinic who are given the pregnancy test are pregnant, but 30% are not.

The test's manufacturer indicates that if a woman is pregnant, the test will be positive 92% of the time.

But if a woman is not pregnant, the test will be positive only 2% of the time and will be negative 98% of the time.

a. Find the probability that a woman at this clinic is pregnant **and** tests positive.

b. Find the probability that a woman at this clinic is actually pregnant **given that** she tests positive.

9.3 Expected Value

In this section you will learn to:

1. *find the expected value of a discrete probability distribution*
2. *interpret expected value as a long-run average*

An expected gain or loss in a game of chance is called **Expected Value**. The concept of expected value is closely related to a *weighted average*. Consider the following situations.

1. Suppose you and your friend play a game that consists of rolling a die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you $4. But if the die shows a 6, you will have to pay him $18.

   Before you play the game you decide to find the expected value. You analyze as follows.

   Since a die will show a number from 1 to 6, with an equal probability of 1/6, your chance of winning $1 is 1/6, winning $2 is 1/6, and so on up to the face value of 5. But if the die shows a 6, you will lose $18. You write the expected value.

   $$E = \$1(1/6) + \$2(1/6) + \$3(1/6) + \$4(1/6) + \$5(1/6) - \$18(1/6) = -\$.50$$

   This means that every time you play this game, you can expect to lose 50 cents. In other words, if you play this game 100 times, theoretically you will lose $50. Obviously, it is not to your interest to play.

2. Suppose of the ten quizzes you took in a course, on eight quizzes you scored 80, and on two you scored 90. You wish to find the average of the ten quizzes.
   The average is

   $$A = \frac{(80)(8) + (90)(2)}{10} = (80)\frac{8}{10} + (90)\frac{2}{10} = 82$$

   It should be observed that it would be incorrect to take the average of 80 and 90 because you scored 80 on eight quizzes, and 90 on only two of them. Therefore, you take a "weighted average" of 80 and 90. That is, the average of 8 parts of 80 and 2 parts of 90, which is 82.

In the first situation, to find the expected value, we multiplied each payoff by the probability of its occurrence, and then added up the amounts calculated for all possible cases. In the second example, if we consider our test score a payoff, we did the same. This leads us to the following definition.

Expected Value

If an experiment has the following probability distribution,

Payoff	x_1	x_2	x_3	. . .	x_n
Probability	$p(x_1)$	$p(x_2)$	$p(x_3)$	. . .	$p(x_n)$

then the expected value of the experiment is

$$\textbf{Expected Value} = x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + \cdots + x_np(x_n)$$

◆***Example 1*** In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and 10% of the families have no children. What is the expected number of children to a family?

Solution: We list the information in the following table.

Number of children	3	2	1	0
Probability	.10	.60	.20	.10

Expected Value = $x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4)$

E = 3(.10) + 2(.60) + 1(.20) + 0(.10) = 1.7

So on average, there are 1.7 children to a family.

◆***Example 2*** To sell an average house, a real estate broker spends $1200 for advertisement expenses. If the house sells in three months, the broker makes $8,000. Otherwise, the broker loses the listing. If there is a 40% chance that the house will sell in three months, what is the expected payoff for the real estate broker?

Solution: The broker makes $8,000 with a probability of .40, but he loses $1200 whether the house sells or not.

E = ($8000)(.40) – ($1200) = $2,000.

Alternatively, the broker makes $(8000 – 1200) with a probability of .40, but loses $1200 with a probability of .60. Therefore,

E = ($6800)(.40) – ($1200)(.60) = $2,000.

◆***Example 3*** In a town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a cold day the attendance is 40,000, and on a stormy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the days will be sunny, 50% of the days will be cold, and 20% days will be stormy, what is the expected attendance for a single game?

Solution: Using the expected value formula, we get

E = (60,000)(.30) + (40,000)(.50) + (30,000)(.20) = 44,000.

◆***Example 4*** A lottery consists of choosing 6 numbers from a total of 51 numbers. The person who matches all six numbers wins $2 million. If the lottery ticket costs $1, what is the expected payoff?

Solution: Since there are 51C6 = 18,009,460 combinations of six numbers from a total of 51 numbers, the chance of choosing the winning number is 1 out of 18,009,460.

So the expected payoff is: $E = (\$2 \text{ million})\left(\frac{1}{18009460}\right) - \$1 = -\$0.89$

This means that every time a person spends $1 to buy a ticket, he or she can expect to lose 89 cents.

SECTION 9.3 PROBLEM SET: EXPECTED VALUE

Do the following problems using the expected value concepts learned in this section,

1) You are about to make an investment which gives you a 30% chance of making $60,000 and 70% chance of losing $ 30,000. Should you invest? Explain.	2) In a town, 40% of the men and 30% of the women are overweight. If the town has 46% men and 54% women, what percent of the people are overweight?
3) A game involves rolling a Korean die (4 faces). If a one, two, or three shows, the player receives the face value of the die in dollars, but if a four shows, the player is obligated to pay $4. What is the expected value of the game?	4) A game involves rolling a single die. One receives the face value of the die in dollars. How much should one be willing to pay to roll the die to make the game fair?
5) In a European country, 20% of the families have three children, 40% have two children, 30% have one child, and 10% have no children. On average, how many children are there to a family?	6) A game involves drawing a single card from a standard deck. One receives 60 cents for an ace, 30 cents for a king, and 5 cents for a red card that is neither an ace nor a king. If the cost of each draw is 10 cents, should one play? Explain.

SECTION 9.3 PROBLEM SET: EXPECTED VALUE

7) Hillview Church plans to raise money by raffling a television worth $500. A total of 3000 tickets are sold at $1 each. Find the expected value of the winnings for a person who buys a ticket in the raffle.	8) During her four years at college, Niki received A's in 30% of her courses, B's in 60% of her courses, and C's in the remaining 10%. If A = 4, B = 3, and C = 2, find her grade point average.
9) Attendance at a Stanford football game depends upon which team Stanford is playing against. If the game is against U. C. Berkeley,attend-ance will be 70,000; if it is against another California team, it will be 40,000; and if it is against an out of state team, it will be 30,000. If the probability of playing against U. C. Berkeley is 10%, against a California team 50% and against an out of state team 40%, how many fans are expected to attend a game?	10) A Texas oil drilling company has determined that it costs $25,000 to sink a test well. If oil is hit, the revenue for the company will be $500,000. If natural gas is found, the revenue will be $150,000. If the probability of hitting oil is 3% and of hitting gas is 6%, find the expected value of sinking a test well.
11) A $1 lottery ticket offers a grand prize of $10,000; 10 runner-up prizes each pay $1000; 100 third-place prizes each pay $100; and 1,000 fourth-place prizes each pay $10. Find the expected value of entering this contest if 1 million tickets are sold.	12) Assume that for the next heavyweight fight the odds of current champion winning are 15 to 2. A gambler bets $10 that the current champion will lose. If current champion loses, how much can the gambler hope to receive?

SECTION 9.3 PROBLEM SET: EXPECTED VALUE

13) In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.

a. Find the average number of children per household

b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?

14) At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.

a. What percent of students take 5 courses?

b. Find the average number of courses that students take.

.

9.4 Probability Using Tree Diagrams

In this section you will learn to:

1. *use probability trees to organize information in probability problems*
2. *use probability trees to calculate probabilitlies*

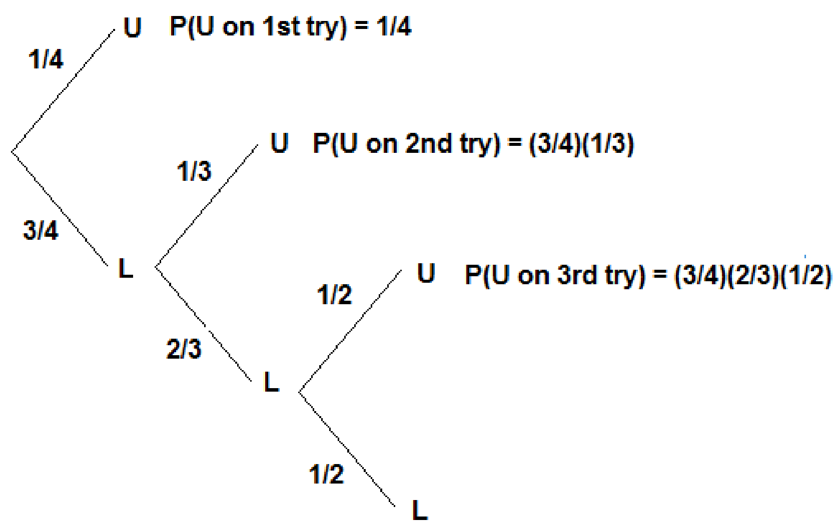
As we have already seen, tree diagrams play an important role in solving probability problems. A tree diagram helps us not only visualize, but also list all possible outcomes in a systematic fashion. Furthermore, when we list various outcomes of an experiment and their corresponding probabilities on a tree diagram, we gain a better understanding of when probabilities are multiplied and when they are added.

The meanings of the words *and* and *or* become clear when we learn to multiply probabilities horizontally across branches, and add probabilities vertically down the tree.

Although tree diagrams are not practical in situations where the possible outcomes become large, they are a significant tool in breaking the problem down in a schematic way. We consider some examples that may seem difficult at first, but with the help of a tree diagram, they can easily be solved.

◆***Example 1*** A person has four keys and only one key fits to the lock of a door. What is the probability that the locked door can be unlocked in at most three tries?

Solution: Let U be the event that the door has been unlocked and L be the event that the door has not been unlocked. We illustrate with a tree diagram.



The probability of unlocking the door in the first try = 1/4

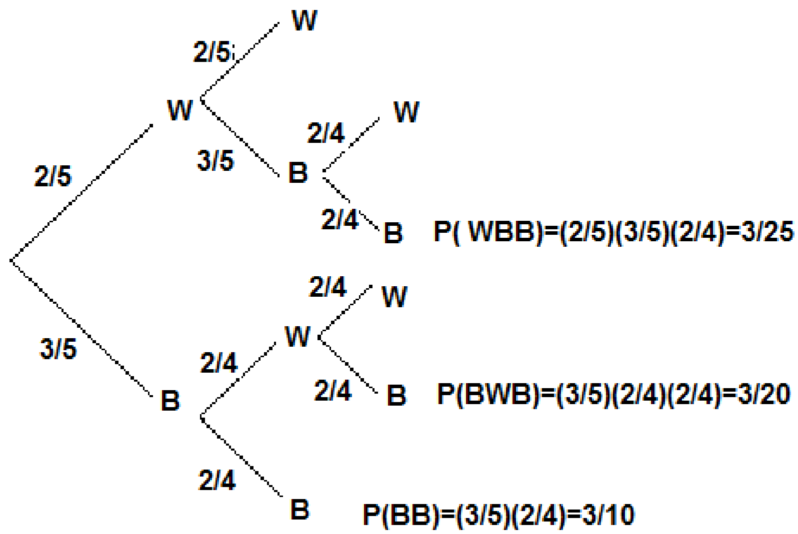
The probability of unlocking the door in the second try = (3/4)(1/3) = 1/4

The probability of unlocking the door in the third try = (3/4)(2/3)(1/2) = 1/4

Therefore, the probability of unlocking the door in at most three tries = 1/4 + 1/4 + 1/4 = 3/4.

◆***Example 2*** A jar contains 3 black and 2 white marbles. We continue to draw marbles one at a time until two black marbles are drawn. If a white marble is drawn, the outcome is recorded and the marble is put back in the jar before drawing the next marble. What is the probability that we will get exactly two black marbles in at most three tries?

Solution: We illustrate using a tree diagram.



The probability that we will get two black marbles in the first two tries is listed adjacent to the lowest branch, and it = 3/10.

The probability of getting first black, second white, and third black = 3/20.

Similarly, the probability of getting first white, second black, and third black = 3/25.

Therefore, the probability of getting exactly two black marbles in at most three tries = 3/10 + 3/20 + 3/25 = 57/100.

◆***Example 3*** A circuit consists of three resistors: resistor R_1, resistor R_2, and resistor R_3, joined in a series. If one of the resistors fails, the circuit stops working. The probabilities that resistors R_1, R_2, or R_3 will fail are .07, .10, and .08, respectively.
Find the probability that at least one of the resistors will fail?

Solution: The probability that at least one of the resistors fails = 1 – none of the resistors fails.

It is quite easy to find the probability of the event that none of the resistors fails. We don't even need to draw a tree because we can visualize the only branch of the tree that assures this outcome.

The probabilities that R_1, R_2, R_3 will not fail are .93, .90, and .92 respectively. Therefore, the probability that none of the resistors fails = (.93)(.90)(.92) = .77.

Thus, the probability that at least one of them will fail = 1 – .77 = .23.

SECTION 9.4 PROBLEM SET: PROBABILITY USING TREE DIAGRAMS

Use a tree diagram to solve the following problems.

1) Suppose you have five keys and only one key fits to the lock of a door. What is the probability that you can open the door in at most three tries?	2) A coin is tossed until a head appears. What is the probability that a head will appear in at most three tries?
3) A basketball player has an 80% chance of making a basket on a free throw. If he makes the basket on the first throw, he has a 90% chance of making it on the second. However, if he misses on the first try, there is only a 70% chance he will make it on the second. If he gets two free throws, what is the probability that he will make at least one of them?	4) You are to play three games. In the first game, you draw a card, and you win if the card is a heart. In the second game, you toss two coins, and you win if one head and one tail are shown. In the third game, two dice are rolled and you win if the sum of the dice is 7 or 11. What is the probability that you win all three games? What is the probability that you win exactly two games?

SECTION 9.4 PROBLEM SET: PROBABILITY USING TREE DIAGRAMS

Use a tree diagram to solve the following problems.

5) John's car is in the garage, and he has to take a bus to get to school. He needs to make all three connections on time to get to his class. If the chance of making the first connection on time is 80%, the second 80%, and the third 70%, what is the chance that John will make it to his class on time?	6) For a real estate exam the probability of a person passing the test on the first try is .70. The probability that a person who fails on the first try will pass on each of the successive attempts is .80. What is the probability that a person passes the test in at most three attempts?
7) On a Christmas tree with lights, if one bulb goes out, the entire string goes out. If there are twelve bulbs on a string, and the probability of any one going out is .04, what is the probability that the string will not go out? \	8) The Long Life Light Bulbs claims that the probability that a light bulb will go out when first used is 15%, but if it does not go out on the first use the probability that it will last the first year is 95%, and if it lasts the first year, there is a 90% probability that it will last two years. Find the probability that a new bulb will last 2 years.

SECTION 9.4 PROBLEM SET: PROBABILITY USING TREE DIAGRAMS

9) A die is rolled until an ace (1) shows. What is the probability that an ace will show on the fourth try?	10) If there are four people in a room, what is the probability that no two have the same birthday?
11) Dan forgets to set his alarm 60% of the time. If he hears the alarm, he turns it off and goes back to sleep 20% of the time, and even if he does wake up on time, he is late getting ready 30% of the time. What is the probability that Dan will be late to school?	12) It has been estimated that 20% of the athletes take some type of drugs. A drug test is 90% accurate, that is, the probability of a false-negative is 10%. Furthermore, for this test the probability of a false-positive is 20%. If an athlete tests positive, what is the probability that he is a drug user?

SECTION 9.5 PROBLEM SET: CHAPTER REVIEW

1) A coin is tossed five times. Find the following

   a) P(2 heads and 3 tails)
   b) P(at least 4 tails)

2) A dandruff shampoo helps 80% of the people who use it. If 10 people apply this shampoo to their hair, what is the probability that 6 will be dandruff free?

3) A baseball player has a .250 batting average. What is the probability that he will have 2 hits in 4 times at bat?

4) Suppose that 60% of the voters in California intend to vote Democratic in the next election. If we choose five people at random, what is the probability that at least four will vote Democratic?

5) A basketball player has a .70 chance of sinking a basket on a free throw. What is the probability that he will sink at least 4 baskets in six shots?

6) During an archery competition, Stan has a 0.8 chance of hitting a target. If he shoots three times, what is the probability that he will hit the target all three times?

7) A company finds that one out of four new applicants overstate their work experience. If ten people apply for a job at this company, what is the probability that at most two will overstate their work experience?

8) A missile has a 70% chance of hitting a target. How many missiles should be fired to make sure that the target is destroyed with a probability of .99 or more?

9) Jar I contains 4 red and 5 white marbles, and Jar II contains 2 red and 4 white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram and find,

   a) P(Marble is red)
   b) P(It is white given that it came from Jar II)
   c) P(It came from Jar II knowing that the marble drawn is white)

10) Suppose a test is given to determine if a person is infected with HIV. If a person is infected with HIV, the test will detect it in 90% of the cases; and if the person is not infected with HIV, the test will show a positive result 3% of the time. If we assume that 2% of the population is actually infected with HIV, what is the probability that a person obtaining a positive result is actually infected with HIV?

11) A car dealer’s inventory consists of 70% cars and 30% trucks. 20% of the cars and 10% of the trucks are used vehicles. If a vehicle chosen at random is used, find the probability that it is a car.

12) Two machines make all the products in a factory, with the first machine making 30% of the products and the second 70%. The first machine makes defective products 3% of the time and the second machine 5% of the time.

    a) Overall what percent of the products made are defective?
    b) If a defective product is found, what is the probability that it was made on the second machine?
    c) If it was made on the second machine, what is the probability that it is defective?

13) An instructor in a finite math course estimates that a student who does his homework has a 90% of chance of passing the course, while a student who does not do the homework has only a 20% chance of passing the course. It has been determined that 60% of the students in a large class do their homework.

    a) What percent of all the students will pass?
    b) If a student passes, what is the probability that he did the homework?

SECTION 9.5 PROBLEM SET: CHAPTER REVIEW

14) Cars are produced at three factories. Factory I produces 10% of the cars and it is known that 2% are defective. Factory II produces 20% of the cars and 3% are defective. Factory III produces 70% of the cars and 4% of those are defective. A car is chosen at random. Find the following probabilities:

    a) P(The car is defective) b) P(The car came from Factory III | the car is defective)

15) A stock has a 50% chance of a 10% gain, a 30% chance of no gain, and otherwise it will lose 8%. Find the expected return.

16) A game involves rolling a pair of dice. One receives the sum of the face value of both dice in dollars. How much should one be willing to pay to roll the dice to make the game fair?

17) A roulette wheel consists of numbers 1 through 36, 0, and 00. If the wheel shows an odd number you win a dollar, otherwise you lose a dollar. If you play the game ten times, what is your expectation?

18) A student takes a 100-question multiple-choice exam in which there are four choices to each question. If the student is just guessing the answers, what score can he expect?

19) Mr. Shaw invests 50% of his money in stocks, 30% in mutual funds, and the remaining 20% in bonds. If the annual yield from stocks is 10%, from mutual funds 12%, and from bonds 7%, what percent return can Mr. Shaw expect on his money?

20) An insurance company is planning to insure a group of surgeons against medical malpractice. Its research shows that two surgeons in every fifteen are involved in a medical malpractice suit each year where the average award to the victim is $450,000. How much minimum annual premium should the insurance company charge each doctor?

21) In an evening finite math class of 30 students, it was discovered that 5 students were of age 20, 8 students were about 25 years old, 10 students were close to 30, 4 students were 35, 2 students were 40 and one student 55. What is the average age of a student in this class?

22) Jar I contains 4 marbles of which one is red, and Jar II contains 6 marbles of which 3 are red. Katy selects a jar and then chooses a marble. If the marble is red, she gets paid 3 dollars, otherwise she loses a dollar. If she plays this game ten times, what is her expected payoff?

23) Jar I contains 1 red and 3 white, and Jar II contains 2 red and 3 white marbles. A marble is drawn from Jar I and put in Jar II. Now if one marble is drawn from Jar II, what is the probability that it is a red marble?

24) Let us suppose there are three traffic lights between your house and the school. The chance of finding the first light green is 60%, the second 50%, and the third 30%. What is the probability that on your way to school, you will find at least two lights green?

25) Sonya has just earned her law degree and is planning to take the bar exam. If her chance of passing the bar exam is 65% on each try, what is the probability that she will pass the exam in at least three tries?

26) Every time a particular baseball player is at bat, his probability of getting a hit is .3, his probability of walking is .1, and his probability of being struck out is .4. If he is at bat three times, what is the probability that he will get two hits and one walk?

27) Jar I contains 4 marbles of which none are red, and Jar II contains 6 marbles of which 4 are red. Juan first chooses a jar and then from it he chooses a marble. After the chosen marble is replaced, Mary repeats the same experiment. What is the probability that at least one of them chooses a red marble?

28) Andre and Pete are two tennis players with equal ability. Andre makes the following offer to Pete: We will not play more than four games, and anytime I win more games than you, I am declared a winner and we stop. Draw a tree diagram and determine Andre's probability of winning.

Chapter 10: Markov Chains

In this chapter, you will learn to:

1. *Write transition matrices for Markov Chain problems.*
2. *Explore some ways in which Markov Chains models are used in business, finance, public health and other fields of application*
3. *Find the long term trend for a Regular Markov Chain.*
4. *Solve and interpret Absorbing Markov Chains.*

10.1 Introduction to Markov Chains

In this chapter, you will learn to:

1. *Write transition matrices for Markov Chain problems.*
2. *Use the transition matrix and the initial state vector to find the state vector that gives the distribution after a specified number of transitions.*

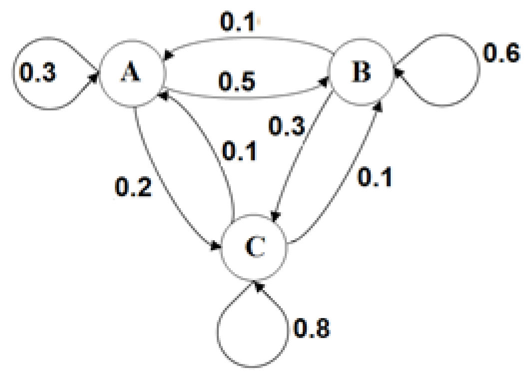
We will now study stochastic processes, experiments in which the outcomes of events depend on the previous outcomes; stochastic processes involve random outcomes that can be described by probabilities. Such a process or experiment is called a **Markov Chain** or **Markov process**. The process was first studied by a Russian mathematician named Andrei A. Markov in the early 1900s.

About 600 cities worldwide have bike share programs. Typically a person pays a fee to join a the program and can borrow a bicycle from any bike share station and then can return it to the same or another system. Each day, the distribution of bikes at the stations changes, as the bikes get returned to different stations from where they are borrowed.

For simplicity, let's consider a very simple bike share program with only 3 stations: A, B, C. Suppose that all bicycles must be returned to the station at the end of the day, so that each day there is a time, let's say midnight, that all bikes are at some station, and we can examine all the stations at this time of day, every day. We want to model the movement of bikes from midnight of a given day to midnight of the next day. We find that over a 1 day period,

- of the bikes borrowed from station A, 30% are returned to station A, 50% end up at station B, and 20% end up at station C.
- of the bikes borrowed from station B, 10% end up at station A, 60% have been returned to station B, and 30% end up at station C
- of the bikes borrowed from station C, 10% end up at station A, 10% end up at station B, and 80% are returned to station C.

We can draw an arrow diagram to show this. The arrows indicate the station where the bicycle was started, called its initial state, and the stations at which it might be located one day later, called the terminal states. The numbers on the arrows show the probability for being in each of the indicated terminal states.



Because our bike share example is simple and has only 3 stations, the arrow diagram, also called a directed graph, helps us visualize the information. But if we had an example with 10, or 20, or more bike share stations, the diagram would become so complicated that it would be difficult to understand the information in the diagram.

We can use a **transition matrix** to organize the information,

Each row in the matrix represents an initial state. Each column represents a terminal state. We will assign the rows in order to stations A, B, C, and the columns in the same order to stations A, B, C. Therefore the matrix must be a square matrix, with the same number of rows as columns. The entry in row 2 column 3, for example, would show the probability that a bike that is initially at station B will be at station C one day later: that entry is 0.30, which is the probability in the diagram for the arrow that points from B to C. We use this the letter T for transition matrix.

$$T = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{array}$$

Looking at the first row that represents bikes initially at station A, we see that 30% of the bikes borrowed from station A are returned to station A, 50% end up at station B, and 20% end up at station C, after one day.

We note some properties of the transition matrix:

- t_{ij} represents the entry in row i column j
- t_{ij} = the probability of moving from state represented by row i to the state represented by row j in a single transition
- t_{ij} is a conditional probability which we can write as:
  t_{ij} = P(next state is the state in column j | current state is the state in row i)
- Each row adds to 1
- All entries are between 0 and 1, inclusive because they are probabilities.
- The transition matrix represents change over one transition period; in this example one transition is a fixed unit of time of one day.

◆***Example 1*** A city is served by two cable TV companies, BestTV and CableCast.

- Due to their aggressive sales tactics, each year 40% of BestTV customers switch to CableCast; the other 60% of BestTV customers stay with BestTV.
- On the other hand, 30% of the CableCast customers switch to Best TV.

The two states in this example are BestTV and CableCast. Express the information above as a transition matrix which displays the probabilities of going from one state into another state.

Solution: The transition matrix is:

		Next year	
		BestTV	CableCast
This year	BestTV	.60	.40
	CableCast	.30	.70

As previously noted, the reader should observe that a transition matrix is always a square matrix because all possible states must have both rows and columns. All entries in a transition matrix are non-negative as they represent probabilities. And, since all possible outcomes are considered in the Markov process, the sum of the row entries is always 1.

With a larger transition matrix, the ideas in Example 1 could be expanded to represent a market with more than 2 cable TV companies. The concepts of brand loyalty and switching between brands demonstrated in the cable TV example apply to many types of products, such as cell phone carriers, brands of regular purchases such as food or laundry detergent, brands major purchases such as cars or appliances, airlines that travelers choose when booking flights, or hotels chains that travelers choose to stay in.

The transition matrix shows the probabilities for transitions between states at two consecutive times. We need a way to represent the distribution among the states at a particular point in time. To do this we use a row matrix called a **state vector**. The state vector is a row matrix that has only one row; it has one column for each state. The entries show the distribution by state at a given point in time. All entries are between 0 and 1 inclusive, and the sum of the entries is 1.

For the bike share example with 3 bike share stations, the state vector is a 1x3 matrix with 1 row and 3 columns. Suppose that when we start observing our bike share program, 30% of the bikes are at station A, 45% of the bikes are at station B, and 25% are at station C. The initial state vector is

$$\begin{array}{cccc} & A & B & C \\ V_0 = & [0.30 & 0.45 & 0.25] \end{array}$$

The subscript 0 indicates that this is the initial distribution, before any transitions occur.

If we want to determine the distribution after one transition, we'll need to find a new state vector that we'll call V_1. The subscript 1 indicates this is the distribution after 1 transition has occurred.

We find V_1 by multiplying V_0 by the transition matrix T, as follows:

$$V_1 = V_0T$$

$$= \begin{bmatrix} 0.30 & 0.45 & 0.25 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} .30(.3)+.45(.1)+.25(.1) & .30(.5)+.45(.6)+.25(.1) & .30(.2)+.45(.3)+.25(.8) \end{bmatrix}$$

$$= \begin{bmatrix} .16 & .445 & .395 \end{bmatrix}$$

After 1 day (1 transition), 16 % of the bikes are at station A, 44.5 % are at station B and 39.5% are at station C.

We showed the step by step work for the matrix multiplication above. In the future we'll generally use technology, such as the matrix capabilities of our calculator, to perform any necessary matrix multiplications, rather than showing the step by step work

Suppose now that we want to know the distribution of bicycles at the stations after two days. We need to find V_2, the state vector after two transitions. To find V_2 , we multiply the state vector after one transition V_1 by the transition matrix T.

$$V_2 = V_1T = \begin{bmatrix} .16 & .445 & .395 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} = \begin{bmatrix} .132 & .3865 & .4815 \end{bmatrix}$$

We note that $V_1 = V_0T$, so $V_2 = V_1T = (V_0T)T = V_0T^2$

This gives an equivalent method to calculate the distribution of bicycles on day 2:

$$V_2 = V_0T^2 = \begin{bmatrix} 0.30 & 0.45 & 0.25 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0.30 & 0.45 & 0.25 \end{bmatrix} \begin{bmatrix} 0.16 & 0.47 & 0.37 \\ 0.12 & 0.44 & 0.44 \\ 0.12 & 0.19 & 0.69 \end{bmatrix}$$

$$V_2 = V_0T^2 = \begin{bmatrix} .132 & .3865 & .4815 \end{bmatrix}$$

After 2 days (2 transitions), 13.2 % of the bikes are at station A, 38.65 % are at station B and 48.15% are at station C.

We need to examine the following: What is the meaning of the entries in the matrix T^2?

$$T^2 = TT = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.16 & 0.47 & 0.37 \\ 0.12 & 0.44 & 0.44 \\ 0.12 & 0.19 & 0.69 \end{bmatrix}$$

The entries in T^2 tell us the probability of a bike being at a particular station after two transitions, given its initial station.

- Entry t_{13} in row 1 column 3 tells us that a bike that is initially borrowed from station A has a probability of 0.37 of being in station C after two transitions.
- Entry t_{32} in row 3 column 2 tells us that a bike that is initially borrowed from station C has a probability of 0. 19 of being in station B after two transitions.

Similarly, if we raise transition matrix T to the nth power, the entries in T^n tells us the probability of a bike being at a particular station after n transitions, given its initial station.

And if we multiply the initial state vector V_0 by T^n, the resulting row matrix $Vn=V_0T^n$ is the distribution of bicycles after n transitions.

◆***Example 2*** Refer to Example 1 for the transition matrix for market shares for subscribers to two cable TV companies.
a. Suppose that today 1/4 of customers subscribe to BestTV and 3/4 of customers subscribe to CableCast. After 1 year, what percent subscribe to each company?
b. Suppose instead that today of 80% of customers subscribe to BestTV and 20% subscribe to CableCast. After 1 year, what percent subscribe to each company?

Solution: a.The initial distribution given by the initial state vector is a 1×2 matrix

$$V_0 = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} .25 & .75 \end{bmatrix}$$

and the transition matrix is

$$T = \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix}$$

After 1 year, the distribution of customers is

$$V_1 = V_0T = \begin{bmatrix} .25 & .75 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} .375 & .625 \end{bmatrix}$$

After 1 year, 37.5% of customers subscribe to BestTV and 62.5% to CableCast.

b. The initial distribution given by the initial state vector $V_0 = \begin{bmatrix} .8 & .2 \end{bmatrix}$. Then

$$V_1 = V_0T = \begin{bmatrix} .8 & .2 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} .54 & .46 \end{bmatrix}$$

In this case, after 1 year, 54% of customers subscribe to BestTV and 46% to CableCast. Note that the distribution after one transition depends on the initial distribution; the distributions in parts (a) and (b) are different because of the different initial state vectors.

◆***Example 3*** Professor Symons either walks to school, or he rides his bicycle. If he walks to school one day, then the next day, he will walk or cycle with equal probability. But if he bicycles one day, then the probability that he will walk the next day is 1/4. Express this information in a transition matrix.

Solution: We obtain the following transition matrix by properly placing the row and column entries. Note that if, for example, Professor Symons bicycles one day, then the probability that he will walk the next day is 1/4, and therefore, the probability that he will bicycle the next day is 3/4.

		Next Day	
		Walk	Bicycle
Initial Day	Walk	1/2	1/2
	Bicycle	1/4	3/4

◆***Example 4*** In Example 3, if it is assumed that the initial day is Monday, write a matrix that gives probabilities of a transition from Monday to Wednesday.

Solution: If today is Monday, then Wednesday is two days from now, representing two transitions. We need to find the square, T^2, of the original transition matrix T, using matrix multiplication.

$$T = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$T^2 = T \times T = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4+1/8 & 1/4+3/8 \\ 1/8+3/16 & 1/8+9/16 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

Recall that we do not obtain T^2 by squaring each entry in matrix T, but obtain T^2 by multiplying matrix T by itself using matrix multiplication.

We represent the results in the following matrix.

		Wednesday	
		Walk	Bicycle
Monday	Walk	3/8	5/8
	Bicycle	5/16	11/16

We interpret the probabilities from the matrix T^2 as follows:

P(Walked Wednesday | Walked Monday) = 3/8.

P(Bicycled Wednesday | Walked Monday) = 5/8.

P(Walked Wednesday | Bicycled Monday) = 5/16.

P(Bicycled Wednesday | Bicycled Monday) = 11/16.

◆***Example 5*** The transition matrix for Example 3 is given below.

		Tuesday	
		Walk	Bicycle
Monday	Walk	1/2	1/2
	Bicycle	1/4	3/4

Write the transition matrix from a) Monday to Thursday, b) Monday to Friday.

Solution: In writing a transition matrix from Monday to Thursday, we are moving from one state to another in three steps. That is, we need to compute T^3.

$$T^3 = \begin{bmatrix} 11/32 & 21/32 \\ 21/64 & 43/64 \end{bmatrix}$$

b) To find the transition matrix from Monday to Friday, we are moving from one state to another in 4 steps. Therefore, we compute T^4.

$$T^4 = \begin{bmatrix} 43/128 & 85/128 \\ 85/256 & 171/256 \end{bmatrix}$$

It is important that the student is able to interpret the above matrix correctly. For example, the entry 85/128, states that if Professor Symons walked to school on Monday, then there is 85/128 probability that he will bicycle to school on Friday.

There are certain Markov chains that tend to stabilize in the long run. We will examine these more deeply later in this chapter. The transition matrix we have used in the above example is just such a Markov chain. The next example deals with the long term trend or steady-state situation for that matrix.

◆***Example 6*** Suppose Professor Symons continues to walk and bicycle according to the transition matrix given in Example 3. In the long run, how often will he walk to school, and how often will he bicycle?

Solution: If we examine higher powers of the transition matrix T, we will find that it stabilizes.

$$T^5 = \begin{bmatrix} .333984 & 666015 \\ .333007 & .666992 \end{bmatrix} \qquad T^{10} = \begin{bmatrix} .33333397 & .66666603 \\ .33333301 & .66666698 \end{bmatrix}$$

$$\text{And} \quad T^{20} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} \qquad \text{and } T^n = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} \text{ for } n > 20$$

The matrix shows that in the long run, Professor Symons will walk to school 1/3 of the time and bicycle 2/3 of the time.

When this happens, we say that the system is in steady-state or state of equilibrium. In this situation, all row vectors are equal. If the original matrix is an n by n matrix, we get n row vectors that are all the same. We call this vector a **fixed probability vector** or the **equilibrium vector** E. In the above problem, the fixed probability vector E is [1/3 2/3]. Furthermore, if the equilibrium vector E is multiplied by the original matrix T, the result is the equilibrium vector E. That is,

$$ET = E \text{ , or} \qquad \begin{bmatrix} 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

SECTION 10.1 PROBLEM SET: INTRODUCTION TO MARKOV CHAINS

1) Is the matrix given below a transition matrix for a Markov chain? Explain.

a) $\begin{bmatrix} .2 & .3 & .5 \\ .3 & -.2 & .9 \\ .3 & .3 & .5 \end{bmatrix}$	b) $\begin{bmatrix} .3 & .3 & .4 \\ .3 & .4 & .4 \\ 0 & 0 & 0 \end{bmatrix}$

2) A survey of American car buyers indicates that if a person buys a Ford, there is a 60% chance that their next purchase will be a Ford, while owners of a GM will buy a GM again with a probability of .80. The buying habits of these consumers are represented in the transition matrix below.

		Next Purchase	
		Ford	GM
Present	Ford	.60	.40
Purchase	GM	.20	.80

Find the following probabilities:

a) The probability that a present owner of a Ford will buy a GM as his next car.	b) The probability that a present owner of a GM will buy a GM as his next car.
c) The probability that a present owner of a Ford will buy a GM as his third car.	d) The probability that a present owner of a GM will buy a GM as his fourth car.

3) Professor Hay has breakfast at Hogee's every morning. He either orders an Egg Scramble, or a Tofu Scramble. He never orders Eggs on two consecutive days, but if he does order Tofu one day, then the next day he can order Tofu or Eggs with equal probability.

a) Write a transition matrix for this problem.	b) If Professor Hay has Tofu on Monday, what is the probability he will have Tofu on Tuesday?
c) If Professor Hay has Eggs on Monday, find the probability he will have Tofu on Wednesday.	d) If Professor Hay has Eggs on Monday, what is the probability he will have Tofu on Thursday?

SECTION 10.1 PROBLEM SET: INTRODUCTION TO MARKOV CHAINS

4) A professional tennis player always hits cross-court or down the line. In order to give himself a tactical edge, he never hits down the line two consecutive times, but if he hits cross-court on one shot, on the next shot he can hit cross-court with .75 probability and down the line with .25 probability.

a) Write a transition matrix for this problem.	b) If the player hit the first shot cross-court, what is the probability that he will hit the third shot down the line?

5) The transition matrix for people voting for candidates from various political parties in an election year is given below. If a person votes for the candidate from one party in an election, that person may vote for the same party in the next election or may switch to vote for a candidate from another party in the next election. Democrats, Republicans, and Independents are denoted by the letters D, R, and I.

$$\begin{array}{ll} & \text{Next Election} \\ & \begin{array}{ccc} \;\;\;\;\;\;\;\;\text{D} & \text{R} & \text{I} \end{array} \\ \begin{array}{l} \\ \text{This} \\ \text{Election} \end{array} \begin{array}{c} \text{D} \\ \text{R} \\ \text{I} \end{array} & \begin{bmatrix} .6 & .3 & .1 \\ .3 & .6 & .1 \\ .2 & .2 & .6 \end{bmatrix} \end{array}$$

Assume there is an election every year so that the transition period is 1 year.

a) Find the probability that a person who votes Democratic in the current election will vote Republican in the next election.	b) Find the probability that a person who votes Democratic in the current election will vote Republican in the election two years from now.
c) Find the probability that a person who votes Republican in the current election will vote Independent in the election two years from now.	d) Find the probability that a person who votes Democratic in the current election will vote independent in the election three years from now.

10.2 Applications of Markov Chains

In this section you will examine some ways in which Markov Chains models are used in business, finance, public health and other fields of application

In the previous section, we examined several applications of Markov chains. Before we proceed further in our investigations of the mathematics of Markov chains in the next sections, we take the time in this section to examine how Markov chains are used in real world applications.

In our bike share program example, we modelled the distribution of the locations of bicycles at bike share stations using a Markov chain. Markov chains have been proposed to model locations of cars distributed among multiple car rental locations for a car rental company, and locations of cars in car share programs. Markov chains models analyze package delivery schedules when packages are transported between several intermediate transport and storage locations on their way to their final destination. In these situations, Markov chains are often one part of larger mathematical models using a combination of other techniques, such as optimization to maximize profit or revenue or minimize cost using linear programming.

In our cable TV example, we modelled market share in a simple example of two cable TV providers. Markov chains can be similarly used in market research studies for many types of products and services, to model brand loyalty and brand transitions as we did in the cable TV model. In the field of finance, Markov chains can model investment return and risk for various types of investments.

Markov chains can model the probabilities of claims for insurance, such as life insurance and disability insurance, and for pensions and annuities. For example, for disability insurance, a much simplified model might include states of healthy, temporarily disabled, permanently disabled, recovered, and deceased; additional refinements could distinguish between disabled policyholders still in the waiting period before collecting benefits and claims actively collecting benefits.

Markov chains have been used in the fields of public health and medicine. Markov chains models of HIV and AIDS include states to model HIV transmission, progression to AIDs, and survival (living with HIV or AIDS) versus death due to AIDS. Comparing Markov chain models of HIV transmission and AIDs progression for various risk groups and ethnic groups can guide public health organizations in developing strategies for reducing risk and managing care for these various groups of people. In general, modeling transmission of various infectious diseases with Markov chains can help in determination of appropriate public health responses to monitor and slow or halt the transmission of these diseases and to determine the most efficient ways to approach treating the disease.

Markov chains have many health applications besides modeling spread and progression of infectious diseases. When analyzing infertility treatments, Markov chains can model the probability of successful pregnancy as a result of a sequence of infertility treatments. Another medical application is analysis of medical risk, such as the role of risk in patient condition following surgery; the Markov chain model quantifies the probabilities of patients progressing between various states of health.

Markov chains are used in ranking of websites in web searches. Markov chains model the probabilities of linking to a list of sites from other sites on that list; a link represents a transition. The Markov chain is analyzed to determine if there is a steady state distribution, or equilibrium, after many transitions. Once equilibrium is identified, the pages with the probabilities in the equilibrium distribution determine the ranking of the webpages. This is a very simplified description of how Google uses Markov chains and matrices to determine "Page rankings" as part of their search algorithms.

Of course, a real world use of such a model by Google would involve immense matrices with thousands of rows and columns. The size of such matrices requires some modifications and use of more sophisticated techniques than we study for Markov chains in this course. However the methods we study form the underlying basis for this concept. It is interesting to note that the term Page ranking does not refer to the fact that webpages are ranked, but instead is named after Google founder Larry Page, who was instrumental in developing this application of Markov chains in the area of web page search and rankings.

Markov chains are also used in quality analysis of cell phone and other communications transmissions. Transition matrices model the probabilities of certain types of signals being transmitted in sequence. Certain sequences of signals are more common and expected, having higher probabilities; on the other hand, other sequences of signals are rare and have low probabilities of occurrence. If certain sequences of signals that are unlikely to occur actually do occur, that might be an indication of errors in transmissions; Markov chains help identify the sequences that represent likely transmission errors.

SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Questions 1-2 refer to the following:

Reference: Bart Sinclair, Machine Repair Model. OpenStax CNX. Jun 9, 2005 Creative Commons Attribution License 1.0. Download for free at http://cnx.org/contents/56f1bed0-bd34-4c28-a2ec-4a3f9ded8e18@3. This material has been modified by Roberta Bloom, as permitted under that license.

A Markov chain can be used to model the status of equipment, such as a machine used in a manufacturing process. Suppose that the possible states for the machine are

Idle & awaiting work (I) Working on a job/task (W) Broken (B) In Repair (R)

The machine is monitored at regular intervals to determine its status; for ease of interpretation in this problem, we assume the status is monitored every hour. The transition matrix is shown below.

$$T = \begin{array}{c} \\ I \\ W \\ B \\ R \end{array} \begin{array}{c} \begin{array}{cccc} I & W & B & R \end{array} \\ \begin{bmatrix} 0.05 & 0.93 & 0.02 & 0 \\ 0.10 & 0.86 & 0.04 & 0 \\ 0 & 0 & 0.80 & 0.20 \\ 0.5 & 0.1 & 0 & 0.4 \end{bmatrix} \end{array}$$

1. Use the transition matrix to identify the following probabilities concerning the state of the machine one hour from now	2. Perform the appropriate calculations using the transition matrix to find the following probabilities concerning the state of the machine three hours from now.
a) Find the probability that the machine is working on a job one hour from now if the machine is idle now.	a) Find the probability that the machine is working on a job three hours from now if the machine is idle now.
b) Find the probability that the machine is idle one hour from now if the machine is working on a job now.	b) Find the probability that the machine is idle three hours from now if the machine is working on a job now.
c) Find the probability that the machine is working on a job one hour from now if the machine is being repaired now.	c) Find the probability that the machine is working on a job three hours from now if the machine is being repaired now.
d) Find the probability that the machine is being repaired in one hour if it is broken now.	d) Find the probability that the machine is being repaired in three hours if it is broken now.

SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Questions 3-4 refer to the following description of how a Markov chain might be used to "train" a computer to generate music.

Teaching a computer music theory so that it can create music would be an extremely tedious task. You would have to teach chord structure, different musical styles, and so on. What if you could give the program examples of pieces you considered to be music and ask it, "write something like that for me." This is essentially how our Markov chain would work. The principle behind Markov chains in music is to generate a probability table to determine what note should come next. By feeding the program an example piece of music, the program can analyze the piece and create a probability table to determine which notes are more likely follow a given note. With the probability transition matrix one can generate random notes that still has some musical structure to it. By constructing a similar matrix for beats or note durations, one can complete a Markov chain model for music generation.

The transition matrix below provides an example. The states are the notes A, A#,B,C,D,E,F,G,G#. The matrix shows the probability of the next note (column state), given the current note (row state).

To generate computer created music, a computer program would randomly select the next note based on the previous note and the probabilities given in the transition matrix.

$$T = \begin{array}{c} \\ A \\ A\# \\ B \\ C \\ D \\ E \\ F \\ G \\ G\# \end{array} \begin{array}{c} \begin{array}{ccccccccc} A & A\# & B & C & D & E & F & G & G\# \end{array} \\ \begin{bmatrix} 0.2 & 0 & 0.1 & 0 & 0.1 & 0.1 & 0 & 0.3 & 0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.1 & 0.3 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0.5 & 0 & 0 & 0 \\ 0.2 & 0.1 & 0 & 0.1 & 0 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0 & 0.6 & 0 \\ 0.2 & 0 & 0.2 & 0.4 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0.75 & 0 & 0 & 0.25 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3) Find the probability for the next note, given the current note a. P(next note is A | current note is G) = ______ b. P(next note is G | current note is A) = ______ c. P(next note is C | current note is F) = ______ d. P(next note is E | current note is D) = ______	4a) If the current note is A# (A-sharp) what does the transition matrix tell us about the next note? 4b) If the next note is F, what do we know about the current note.

SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Question 5 refers to the following:

Markov chains play an important role in online search.

"PageRank is an algorithm used by Google Search to rank websites in their search engine results. PageRank was named after Larry Page, one of the founders of Google. PageRank is a way of measuring the importance of website pages"
Source: https://en.wikipedia.org/wiki/PageRank under the Creative Commons Attribution-ShareAlike License;

The theory behind PageRank is that pages that are linked to more often are more important and useful; identifying those that are linked to more often about a topic helps identify the pages that should be presented as most pertinent in a search.

In real world search, there are thousands or millions of pages linking together, resulting in huge transition matrices. Because of the size and other properties of these matrices, the mathematics behind PageRank is more sophisticated than the small example we examine here with only four websites. However our example is adequate to convey the main concept of PageRank and its use in search algorithms.

It should be noted that real world search algorithms, PageRank or similar Markov chain ranking schemes are only one of a variety of methods used.

Suppose we have 4 webpages that contains links to each other. We call the pages A, B, C, D.

- From page A, 30% of people link to page B, 50% link to page C, and 20% link to page D
- From page B, 50% of popele link to page A and 50% link to page D
- From page C, 10% of people link to page B, 70% link to page C, and 20% link to page D
- From page D, 20% of people link to page A, 40% to page B, 10% to page C, and 30% link to page D

(In this example, when a page links to itself, it means that a person viewing the page stays at that page and does not link to another page.)

a) Write the transition matrix T	b) Find the probability that a person viewing page C will link to page D next.
c) Find the probability that a person viewing page C will view page D after two links	d) Find the probability that a person viewing page C will view page D after **three** links
e) Find the probability that a person viewing page C will stay at page C and not link to any other page next.	f) Find the probability that a person viewing page C will view page A next

10.3 Regular Markov Chains

In this section, you will learn to:

1. *identify Regular Markov Chains, which have an equilibrium or steady state in the long run*
2. *find the long term equilibrium for a Regular Markov Chain.*

At the end of section 10.1, we examined the transition matrix T for Professor Symons walking and biking to work. As we calculated higher and higher powers of T, the matrix started to stabilize, and finally it reached its **steady-state** or **state of equilibrium**. When that happened, all the row vectors became the same, and we called one such row vector a **fixed probability vector** or an **equilibrium vector** E. Furthermore, we discovered that ET = E.

In this section, we wish to answer the following four questions.

1) Does every Markov chain reach a state of equilibrium? Is there a way to determine if a Markov chain reaches a state of equilibrium?
2) Does the product of an equilibrium vector and its transition matrix always equal the equilibrium vector? That is, does ET = E?
3) Can the equilibrium vector E be found without raising the matrix to higher powers?
4) Does the long term market share distribution for a Markov chain depend on the initial market share?

◆***Question 1*** Does every Markov chain reach the state of equilibrium?

Answer: No. Some Markov chains reach a state of equilibrium but some do not. Some Markov chains transitions do not settle down to a fixed or equilibrium pattern. Therefore we'd like to have a way to identify Markov chains that do reach a state of equilibrium.

One type of Markov chains that do reach a state of equilibrium are called **regular** Markov chains. A Markov chain is said to be a **regular Markov chain** if some power of its transition matrix T has only positive entries.

To determine if a Markov chain is regular, we examine its transition matrix T and powers, T^n, of the transition matrix. If we find any power n for which T^n has only positive entries (no zero entries), then we know the Markov chain is regular and is guaranteed to reach a state of equilibrium in the long run.

Fortunately, we don't have to examine too many powers of the transition matrix T to determine if a Markov chain is regular; we use technology, calculators or computers, to do the calculations. There is a theorem that says that if an nxn transition matrix represents n states, then we need only examine powers T^m up to $m = (n-1)^2 + 1$. If some power of the transition matrix T^m is going to have only positive entries, then that will occur for some power $m \le (n-1)^2 + 1$

For example, if T is a 3x3 transition matrix, then $m = (n-1)^2 + 1 = (3-1)^2 + 1 = 5$.

- If we examine T, T^2, T^3, T^4 and T^5, and find that any of those matrices has only positive entries, then we know T is regular.
- If however, T, T^2, T^3, T^4 and T^5 all have at least one zero entry and none of them have all positive entries, then we can stop checking. All higher powers of T will also have at least one zero entry, and T will not be regular.

◆***Example 1*** Determine whether the following Markov chains are regular.

a. $A = \begin{bmatrix} 0 & 1 \\ .4 & .6 \end{bmatrix}$ b. $B = \begin{bmatrix} 1 & 0 \\ .3 & .7 \end{bmatrix}$

Solution: a). The transition matrix A does not have all positive entries. But it is a regular Markov chain because

$$A^2 = \begin{bmatrix} .40 & .60 \\ .24 & .76 \end{bmatrix}$$

has only positive entries.

b. The matrix B is not a regular Markov chain because every power of B has an entry 0 in the first row, second column position.

$$B = \begin{bmatrix} 1 & 0 \\ .3 & .7 \end{bmatrix} \quad \text{and } B^2 = \begin{bmatrix} 1 & 0 \\ .3 & .7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .51 & .49 \end{bmatrix}$$

Since B is a 2x2 matrix, $m = (2-1)^2+1 = 2$. We've examined B and B^2, and discovered that neither has all positive entries. We don't need to examine any higher powers of B; B is not a regular Markov chain.

In fact, we can show that all 2 by 2 matrices that have a zero in the first row, second column position are not regular. Consider the following matrix M.

$$M = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$$

$$M^2 = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a \cdot a + 0 \cdot b & a \cdot 0 + 0 \cdot c \\ b \cdot a + c \cdot b & b \cdot 0 + c \cdot c \end{bmatrix}$$

Observe that the first row, second column entry, $a \cdot 0 + 0 \cdot c$, will always be zero, regardless of what power we raise the matrix to.

◆***Question 2*** Does the product of an equilibrium vector and its transition matrix always equal the equilibrium vector? That is, does ET = E?

Answer: At this point, the reader may have already guessed that the answer is yes if the transition matrix is a regular Markov chain. We try to illustrate with the following example from section 10.1.

A city is served by two cable TV companies, BestTV and CableCast. Due to their aggressive sales tactics, each year 40% of BestTV customers switch to CableCast; the other 60% of BestTV customers stay with BestTV. On the other hand, 30% of the CableCast customers switch to Best RV and 70% of CableCast customers stay with CableCast.

The transition matrix is given below.

		Next Year	
		BestTV	CableCast
Initial Year	BestTV	.60	.40
	CableCast	.30	.70

If the initial market share for BestTV is 20% and for CableCast is 80%, we'd like to know the long term market share for each company.

Let matrix T denote the transition matrix for this Markov chain, and V_0 denote the matrix that represents the initial market share. Then V_0 and T are as follows:

$$V_0 = \begin{bmatrix} .20 & .80 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix}$$

Since each year people switch according to the transition matrix T, after one year the distribution for each company is as follows:

$$V_1 = V_0\,T = \begin{bmatrix} .20 & .80 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} .36 & .64 \end{bmatrix}$$

After two years, the market share for each company is

$$V_2 = V_1\,T = \begin{bmatrix} .36 & .64 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} .408 & .592 \end{bmatrix}$$

After three years the distribution is

$$V_3 = V_2\,T = \begin{bmatrix} .408 & .592 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} .4224 & .5776 \end{bmatrix}$$

After 20 years the market share are given by $V_{20} = V_0\,T^{20} = \begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$.

After 21 years, $V_{21} = V_0\,T^{21} = \begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$; market shares are stable and did not change.

The market share after 20 years has stabilized to $\begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$. This means that

$$\begin{bmatrix} 3/7 & 4/7 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$$

Once the market share reaches an equilibrium state, it stays the same, that is, ET = E.

This helps us answer the next question.

◆***Question 3*** Can the equilibrium vector E be found without raising the transition matrix T to large powers?

Answer: The answer to the second question provides us with a way to find the equilibrium vector E.

The answer lies in the fact that ET = E.

Since we have the matrix T, we can determine E from the statement ET = E.

Suppose E = $\begin{bmatrix} e & 1-e \end{bmatrix}$, then ET = E gives us

$$\begin{bmatrix} e & 1-e \end{bmatrix} \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = \begin{bmatrix} e & 1-e \end{bmatrix}$$

$$\begin{bmatrix} (.60)e + .30(1-e) & (.40)e + .70(1-e) \end{bmatrix} = \begin{bmatrix} e & 1-e \end{bmatrix}$$

$$\begin{bmatrix} .30e + .30 & -.30e + .70 \end{bmatrix} = \begin{bmatrix} e & 1-e \end{bmatrix}$$

$$.30e + .30 = e$$

$$e = 3/7$$

Therefore, E = $\begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$

As a result of our work in questions 2 and 3, we see that we have a choice of methods to find the equilibrium vector.

Method 1: We can determine if the transition matrix T is regular. If T is regular, we know there is an equilibrium and we can use technology to find a high power of T.

- For the question of what is a sufficiently high power of T, there is no “exact” answer.
- Select a “high power”, such as n=30, or n=50, or n=98. Evaluate T^n on your calculator or with a computer. Check if $T^{n+1} = T^n$. If $T^{n+1} = T^n$ and all the rows of T^n are the same, then we’ve found the equilibrium. The equilibrium vector is one row of T^n. But if you did not find equilibrium yet for a regular Markov chain, try using a higher power of T.

Method 2: We can solve the matrix equation ET=E.

- The disadvantage of this method is that it is a bit harder, especially if the transition matrix is larger than 2x2. However it’s not as hard as it seems, if T is not too large a matrix, because we can use the methods we learned in chapter 2 to solve the system of linear equations, rather than doing the algebra by hand.
- The advantage of solving ET = E as in Method 2 is that it can be used with matrices that are not regular. If a matrix is regular, it is guaranteed to have an equilibrium solution. If a matrix is not regular, then it may or may not have an equilibrium solution, and solving ET = E will allow us to prove that it has an equilibrium solution even if the matrix is not regular.
(In mathematics we say that being a regular matrix is a “sufficient” condition for having an equilibrium, but is not a necessary condition.)

◆***Question 4*** Does the long term market share for a Markov chain depend on the initial market share?

Answer: We will show that the final market share distribution for a Markov chain does not depend upon the initial market share. In fact, one does not even need to know the initial market share distribution to find the long term distribution.

Furthermore, the final market share distribution can be found by simply raising the transition matrix to higher powers.

Consider the initial market share $V_0 = [\ .20 \quad .80\ ]$, and the transition matrix

$T = \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix}$ for BestTV and CableCast in the above example.

Recall we found T^n, for very large n, to be $\begin{bmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{bmatrix}$.

Using our calculators, we can easily verify that for sufficiently large n (we used n = 30),

$$V_0T^n = [\ .20 \quad .80\ ] \begin{bmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{bmatrix} = [\ 3/7 \quad 4/7\ ]$$

No matter what the initial market share, the product is $[\ 3/7 \quad 4/7\ ]$.

If instead the initial share is $W_0 = [\ .10 \quad .90\ ]$, then for sufficiently large n

$$W_0T^n = [\ .10 \quad .90\ ] \begin{bmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{bmatrix} = [\ 3/7 \quad 4/7\ ]$$

For any distribution $A = [\ a \quad 1 - a\ ]$, for example,

$$\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{bmatrix} = [\ 3/7(a) + 3/7(1-a) \quad 4/7(a) + 4/7(1-a)\ ]$$

$$= [\ 3/7 \quad 4/7\ ]$$

It makes sense; the entry $3/7(a) + 3/7(1 - a)$, for example, will always equal 3/7.

◆***Example 2*** Three companies, A, B, and C, compete against each other. The transition matrix T for people switching each month among them is given by the following transition matrix.

		Next Month		
		Company A	Company B	Company C
	Company A	.1	.3	.6
Initial Month	Company B	.6	.2	.2
	Company C	.1	.3	.6

If the initial market share for the companies A, B, and C is **[** .25 .35 .40 **]**, what is the long term distribution?

Solution: Since the long term market share does not depend on the initial market share, we can simply raise the transition market share to a large power and get the distribution.

$$T^{20} = [\ 13/55 \quad 3/11 \quad 27/55\ ]$$

In the long term, Company A has 13/55 (about 23.64%) of the market share, Company B has 3/11 (about 27.27%) of the market share, and Company C has 27/55 (about 49.09%) of the market share.

We summarize as follows:

Regular Markov Chains

A Markov chain is said to be a Regular Markov chain if some power of it has only positive entries.

Let T be a transition matrix for a regular Markov chain.

1. As we take higher powers of T, T^n, as n becomes large, approaches a state of equilibrium.
2. If V_0 is any distribution vector, and E an equilibrium vector, then $V_0T^n = E$.
3. Each row of the equilibrium matrix T^n is a unique equilibrium vector E such that ET = E.
4. The equilibrium distribution vector E can be found by letting ET = E.

SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS

1) Determine whether the following matrices are regular Markov chains.

a) $\begin{bmatrix} 1 & 0 \\ .5 & .5 \end{bmatrix}$	b) $\begin{bmatrix} .6 & .4 \\ 0 & 1 \end{bmatrix}$
c) $\begin{bmatrix} .6 & 0 & .4 \\ .2 & .4 & .4 \\ 0 & 0 & 0 \end{bmatrix}$	d) $\begin{bmatrix} .2 & .4 & .4 \\ .6 & .4 & 0 \\ .3 & .2 & .5 \end{bmatrix}$

2) Company I and Company II compete against each other, and the transition matrix for people switching from Company I to Company II is given below.

		TO Company I	Company II
FROM	Company I	.3	.7
	Company II	.8	.2

a) If the initial market share is 40% for Company I and 60% for Company II, what will the market share be after 3 transitions?	b) If this trend continues, what is the long range expectation for the market?

3) Suppose the transition matrix for the tennis player in Exercise 4 of the last section is as follows, where C denotes the cross-court shots and D denotes down-the-line shots.

		Next Shot C	D
Previous Shot	C	.9	.1
	D	.7	.3

a) If the player hit the first shot cross-court, what is the probability he will hit the fourth shot cross-court?	b) Determine the long term shot distribution.

SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS

4) Professor Hay never orders eggs two days in a row, but if he orders tofu one day, then there is an equal probability that he will order tofu or eggs the next day.

Find the following:

a) If Professor Hay had eggs on Monday, what is the probability that he will have tofu on Friday?	b) Find the long term distribution for breakfast choices for Professor Hay.

5) In a bikeshare program with 3 bike stations, A, B, and C, people can borrow a bicycle at one station and return it to the same station or either of the other two stations. The transition matrix is:

$$T = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{array}$$

Find the following:

a) If a bicycle is initially at station A, what is the probability it will be at station C after 5 days?	b) If the initial distribution of bicycles is 50% at station A, 20% at station B, and 30% at station C, what will be the distribution after 2 days? After 5 days?
c) What will be the eventual long term distribution of bicycles at the stations?	d) If initially the distribution of bicycles at the stations was evenly distributed with one third of the bicycles at each station, will the eventual long term distribution be different than if the initial distribution is as given in part (c)?

SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS

6) Suppose that a country has 3 political parties: the Conservative (C), Liberal (L), and National (N) parties. If a person votes for the candidate from one party in an election, that person may decide to vote for the same party in the next election or may switch to vote for a candidate from another party in the next election. The transition matrix is:

NEXT ELECTION

$$\begin{array}{ll} & \begin{array}{ccc} \quad C & \quad L & \quad N \end{array} \\ \begin{array}{r} \\ \text{THIS} \\ \text{ELECTION} \end{array} \begin{array}{c} C \\ L \\ N \end{array} & \begin{bmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .2 & .6 \end{bmatrix} \end{array}$$

Assume there is an election every year, so the transition time is one year. Find the following.

a) If a person voted for the Liberal party in this election, find the probability that the person votes for the National party in the next election.	b) If a person voted for the National party in this election, find the probability that the person votes for the Conservative party in the election two years from now.
c) If in this election Conservatives received 25% of the votes, Liberals 30% of the votes, and Nationals the remaining 45% of the votes, what is the predicted distribution for the next election?	d) Assuming the current distribution from part (c), what will the distribution be in the election two years from now?
e) Assuming the current distribution from part (c), what will the distribution be in the election three years from now?	f) Determine the long term distribution.

SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS

Question 7 refers to the following:

Markov chains play an important role in online search.

"PageRank is an algorithm used by Google Search to rank websites in their search engine results. PageRank was named after Larry Page, one of the founders of Google. PageRank is a way of measuring the importance of website pages"

Source: https://en.wikipedia.org/wiki/PageRank under the Creative Commons Attribution-ShareAlike License;

The theory behind PageRank is that pages that are linked to more often are more important and useful; identifying those that are linked to more often about a topic helps identify the pages that should be presented as most pertinent in a search.

In real world search, there are thousands or millions of pages linking together, resulting in huge transition matrices. Because of the size and other properties of these matrices, the mathematics behind PageRank is more sophisticated than the small example we examine here with only four websites. However our example is adequate to convey the main concept of PageRank and its use in search algorithms.

It should be noted that real world search algorithms, PageRank or similar Markov chain ranking schemes are only one of a variety of methodsused.

Suppose we have 4 webpages that contains links to each other. We call the pages A, B, C, D.

- From page A, 30% of people link to page B, 50% link to page C, and 20% link to page D
- From page B, 50% of popele link to page A and 50% link to page D
- From page C, 10% of people link to page B, 70% link to page C, and 20% link to page D
- From page D, 20% of people link to page A, 40% to page B, 10% to page C, and 30% link to page D

(In this example, when a page links to itself, it means that a person viewing the page stays at that page and does not link to another page.)

a)Write the transition matrix.	b) T is a 4x4 matrix, n= 4 states. Use the formula $m = (n-1)^2 + 1$ to find the highest power m that we need to check to determine if T is a regular Markov chain.
c) Is this a regular Markov chain? Explain how you determined that.	d) Find the equilibrium vector and write a sentence summarizing the long term distribution of visits to these sites based on this model.
e) In the equilibrium vector, the state with the highest probability has the highest "Page-rank" and as the probabilities decrease, the ranking decreases. Indicate the order of the ranking, from highest Page rank to lowest Page rank, of these 4 pages.	

10.4 Absorbing Markov Chains

In this section you will learn to:

1. *identify absorbing states and absorbing Markov chains*
2. *solve and interpret absorbing Markov chains.*

In this section, we will study a type of Markov chain in which when a certain state is reached, it is impossible to leave that state. Such states are called **absorbing states**, and a Markov Chain that has at least one such state is called an **Absorbing Markov chain.**

Suppose you have the following transition matrix.

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} .1 & .3 & .6 \\ 0 & 1 & 0 \\ .3 & .2 & .5 \end{bmatrix} \end{array}$$

The state S_2 is an absorbing state, because the probability of moving from state S_2 to state S_2 is 1. Which is another way of saying that if you are in state S_2, you will remain in state S_2.

In fact, this is the way to identify an absorbing state. If the probability in row i and column i , p_{ii}, is 1, then state S_i is an absorbing state.

◆***Example 1*** Consider transition matrices A, B, C for Markov chains shown below. Which of the following Markov chains have an absorbing state?

$$A = \begin{bmatrix} .3 & .7 & 0 \\ 0 & 1 & 0 \\ .2 & .3 & .5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} .1 & .3 & .4 & .2 \\ 0 & .2 & .1 & .7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$A = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} .3 & .7 & 0 \\ 0 & 1 & 0 \\ .2 & .3 & .5 \end{bmatrix} \end{array}$$ has S_2 as an absorbing state.

If we are in state S_2, we can not leave it.
From state S_2, we can not transition to state S_1 or S_3; the probabilities are 0.
The probability of transition from state S_2 to state S_2 is 1.

$$B = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$ does not have any absorbing states.

From state S_1, we always transition to state S_2. From state S_2 we always transition to state S_3. From state S_3 we always transition to state S_1. In this matrix, it is never possible to stay in the same state during a transition.

$$C = \begin{matrix} & S_1 & S_2 & S_3 & S_4 \\ S_1 & .1 & .3 & .4 & .2 \\ S_2 & 0 & .2 & .1 & .7 \\ S_3 & 0 & 0 & 1 & 0 \\ S_4 & 0 & 0 & 0 & 1 \end{matrix}$$ has two absorbing states, S_3 and S_4.

From state S_3, you can only remain in state S_3, and never transition to any other states. Similarly from state S_4, you can only remain in state S_4, and never transition to any other states.

We summarize how to identify absorbing states.

A state S is an absorbing state in a Markov chain in the transition matrix if

- The row for state S has one 1 and all other entries are 0
  AND
- The entry that is 1 is on the main diagonal (row = column for that entry), indicating that we can never leave that state once it is entered.

Next we define an absorbing Markov Chain

A Markov chain is an absorbing Markov Chain if

- It has at least one absorbing state
  AND
- From any non-absorbing state in the Markov chain, it is possible to eventually move to some absorbing state (in one or more transitions).

◆***Example 2*** Consider transition matrices C and D for Markov chains shown below. Which of the following Markov chains is an absorbing Markov Chain?

$$C = \begin{bmatrix} .1 & .3 & .4 & .2 \\ 0 & .2 & .1 & .7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ .2 & .2 & .2 & .2 & .2 \\ 0 & 0 & 0 & .3 & .7 \\ 0 & 0 & 0 & .6 & .4 \end{bmatrix}$$

Solution: C is an absorbing Markov Chain but D is not an absorbing Markov chain.

Matrix C has two absorbing states, S_3 and S_4, and it is possible to get to state S_3 and S_4 from S_1 and S_2.

Matrix D is not an absorbing Markov chain.has two absorbing states, S_1 and S_2, but it is never possible to get to either of those absorbing states from either S_4 or S_5. If you are in state S_4 or $S_{5,}$ you always remain transitioning between states S_4 or S_{5m} and can never get absorbed into either state S_1 or S_2

In the remainder of this section, we'll examine absorbing Markov chains with two classic problems: the random drunkard's walk problem and the gambler's ruin problem. And finally we'll conclude with an absorbing Markov model applied to a real world situation.

DRUNKARD'S RANDOM WALK

In this example we briefly examine the basic ideas concerning absorbing Markov chains.

◆***Example 3*** A man walks along a three block portion of Main St. His house is at one end of the three block section. A bar is at the other end of the three block section. Each time he reaches a corner he randomly either goes forward one block or turns around and goes back one block. If he reaches home or the bar, he stays there. The four states are Home (H), Corner 1(C_1), Corner 2 (C_2) and Bar (B).
Write the transition matrix and identify the absorbing states. Find the probabilities of ending up in each absorbing state depending on the initial state.

Solution: The transition matrix is written below.

$$T = \begin{array}{c} \\ H \\ C_1 \\ C_2 \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cccc} H & C_1 & C_2 & B \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Home and the Bar are absorbing states. If the man arrives home, he does not leave.
If the man arrives at the bar, he does not leave. Since it is possible to reach home or the bar from each of the other two corners on his walk, this is an absorbing Markov chain.

We can raise the transition matrix T to a high power, n. One we find a power T^n that remains stable, it will tell us the probability of ending up in each absorbing state depending on the initial state.

$$T^{90} = \begin{array}{c} \\ H \\ C_1 \\ C_2 \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cccc} H & C_1 & C_2 & B \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad T^{91} = \begin{array}{c} \\ H \\ C_1 \\ C_2 \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cccc} H & C_1 & C_2 & B \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$T^{91} = T^{90}$; the matrix does not change as we continue to examine higher powers.
We see that in the long-run, the Markov chain must end up in an absorbing state.
In the long run, the man must eventually end up at either his home or the bar.

The second row tells us that if the man is at corner C_1, then there is a 2/3 chance he will end up at home and a 1/3 chance he will end up at the bar.

The third row tells us that if the man is at corner C_2, then there is a 1/3 chance he will end up at home and a 2/3 chance he will end up at the bar.

Once he reaches home or the bar, he never leaves that absorbing state.

Note that while the matrix T^n for sufficiently large n has become stable and is not changing, it does not represent an equilibrium matrix. The rows are not all identical, as we found in the regular Markov chains that reached an equilibrium.

We can write a smaller "solution matrix" by retaining only rows that relate to the non-absorbing states and retaining only the columns that relate to the absorbing states.

Then the solution matrix will have rows C_1 and C_2, and columns H and B.
The solution matrix is

$$\text{Solution Matrix:} \quad \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} H & B \end{array} \\ \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

The first row of the solution matrix shows that if the man is at corner C_1, then there is a 2/3 chance he will end up at home and a 1/3 chance he will end up at the bar.

The second row of the solution matrix shows that if the man is at corner C_2, then there is a 1/3 chance he will end up at home and a 2/3 chance he will end up at the bar.

The solution matrix does not show that eventually there is 0 probability of ending up in C1 or C2, or that if you start in an absorbing state H or B, you stay there. The smaller solution matrix assumes that we understand these outcomes and does not include that information.

The next example is another classic example of an absorbing Markov chain. In the next example we examine more of the mathematical details behind the concept of the solution matrix.

GAMBLER'S RUIN PROBLEM

◆***Example 4*** A gambler has $3,000, and she decides to gamble $1,000 at a time at a Black Jack table in a casino in Las Vegas. She has told herself that she will continue playing until she goes broke or has $5,000. Her probability of winning at Black Jack is .40. Write the transition matrix, identify the absorbing states, find the solution matrix, and determine the probability that the gambler will be financially ruined at a stage when she has $2,000.

Solution: The transition matrix is written below. Clearly the state 0 and state 5K are the absorbing states. This makes sense because as soon as the gambler reaches 0, she is financially ruined and the game is over. Similarly, if the gambler reaches $5,000, she has promised herself to quit and, again, the game is over. The reader should note that $p_{00} = 1$, and $p_{55} = 1$.

Further observe that since the gambler bets only $1,000 at a time, she can raise or lower her money only by $1,000 at a time. In other words, if she has $2,000 now, after the next bet she can have $3,000 with a probability of .40 and $1,000 with a probability of .60.

	0	1K	2K	3K	4K	5K
0	1	0	0	0	0	0
1K	.60	0	.40	0	0	0
2K	0	.60	0	.40	0	0
3K	0	0	.60	0	.40	0
4K	0	0	0	.60	0	.40
5K	0	0	0	0	0	1

To determine the long term trend, we raise the matrix to higher powers until all the non-absorbing states are absorbed. This is the called the **solution matrix.**

	0	1K	2K	3K	4K	5K
0	1	0	0	0	0	0
1K	195/211	0	0	0	0	16/211
2K	171/211	0	0	0	0	40/211
3K	135/211	0	0	0	0	76/211
4K	81/211	0	0	0	0	130/211
5K	0	0	0	0	0	1

The solution matrix is often written in the following form, where the non-absorbing states are written as rows on the side, and the absorbing states as columns on the top.

	0	5K
1K	195/211	16/211
2K	171/211	40/211
3K	135/211	76/211
4K	81/211	130/211

The table lists the probabilities of getting absorbed in state 0 or state 5K starting from any of the four non-absorbing states. For example, if at any instance the gambler has $3,000, then her probability of financial ruin is 135/211 and her probability reaching 5K is 76/211.

◆***Example 5*** Solve the Gambler's Ruin Problem of Example 4 without raising the matrix to higher powers, and determine the number of bets the gambler makes before the game is over.

Solution: In solving absorbing states, it is often convenient to rearrange the matrix so that the rows and columns corresponding to the absorbing states are listed first. This is called the **Canonical form.** The transition matrix of Example 1 in the canonical form is listed below.

	0	5K	1K	2K	3K	4K
0	1	0	0	0	0	0
5K	0	1	0	0	0	0
1K	.60	0	0	.40	0	0
2K	0	0	.60	0	.40	0
3K	0	0	0	.60	0	.40
4K	0	.40	0	0	.60	0

The canonical form divides the transition matrix into four sub-matrices as listed below.

	Absorbing	Non-absorbing
Absorbing states	I_n	O
Non-absorbing states	A	B

The matrix $F = (I_n - B)^{-1}$ is called the fundamental matrix for the absorbing Markov chain, where I_n is an identity matrix of the same size as B. The i, j-th entry of this matrix tells us the average number of times the process is in the non-absorbing state j before absorption if it started in the non-absorbing state i.

The matrix $F = (I_n - B)^{-1}$ for our problem is listed below.

F =

	1K	2K	3K	4K
1K	1.54	.90	.47	.19
2K	1.35	2.25	1.18	.47
3K	1.07	1.78	2.25	.90
4K	.64	1.07	1.35	1.54

You can use your calculator, or a computer, to calculate matrix F.

The Fundamental matrix F helps us determine the average number of games played before absorption.

According to the matrix, the entry 1.78 in the row 3, column 2 position says that the gambler will play the game 1.78 times before she goes from $3K to $2K. The entry 2.25 in row 3, column 3 says that if the gambler now has $3K, she will have $3K on the average 2.25 times before the game is over.

We now address the question of how many bets will she have to make before she is absorbed, if the gambler begins with $3K?

If we add the number of games the gambler plays in each non-absorbing state, we get the average number of games before absorption from that state. Therefore, if the gambler starts with $3K, the average number of Black Jack games she will play before absorption is

$$1.07 + 1.78 + 2.25 + .90 = 6.0$$

That is, we expect the gambler will either have $5,000 or nothing on the 7th bet.

Lastly, we find the solution matrix without raising the transition matrix to higher powers. The matrix FA gives us the solution matrix.

$$FA = \begin{bmatrix} 1.54 & .90 & .47 & .19 \\ 1.35 & 2.25 & 1.18 & .47 \\ 1.07 & 1.78 & 2.25 & .90 \\ .64 & 1.07 & 1.35 & 1.54 \end{bmatrix} \begin{bmatrix} .6 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & .4 \end{bmatrix} = \begin{bmatrix} .92 & .08 \\ .81 & .19 \\ .64 & .36 \\ .38 & .62 \end{bmatrix}$$

which is the same as the following matrix we obtained by raising the transition matrix to higher powers.

$$\begin{array}{c} \\ 1K \\ 2K \\ 3K \\ 4K \end{array} \begin{array}{c} \begin{array}{cc} 0 & 5K \end{array} \\ \begin{bmatrix} 195/211 & 16/211 \\ 171/211 & 40/211 \\ 135/211 & 76/211 \\ 81/211 & 130/211 \end{bmatrix} \end{array}$$

◆***Example 6*** At a professional school, students need to take and pass an English writing/speech class in order to get their professional degree. Students must take the class during the first quarter that they enroll. If they do not pass the class they take it again in the second semester. If they fail twice, they are not permitted to retake it again, and so would be unable to earn their degree.

Students can be in one of 4 states: passed the class (P), enrolled in the class for the first time (C), retaking the class (R) or failed twice and can not retake (F). Experience shows 70% of students taking the class for the first time pass and 80% of students taking the class for the second time pass.

Write the transition matrix and identify the absorbing states.
Find the probability of being absorbed eventually in each of the absorbing states.

Solution: The absorbing states are P (pass) and F (fail repeatedly and can not retake).
The transition matrix T is shown below.

$$T = \begin{array}{c} \\ P \\ C \\ R \\ F \end{array} \begin{array}{c} \begin{array}{cccc} P & C & R & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .7 & 0 & .3 & 0 \\ .8 & 0 & 0 & .2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

If we raise the transition matrix T to a high power, we find that it remains stable and gives us the long-term probabilities of ending up in each of the absorbing states.

$$T^{30} = \begin{array}{c} \\ P \\ C \\ R \\ F \end{array} \begin{array}{c} \begin{array}{cccc} P & C & R & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .94 & 0 & 0 & .06 \\ .8 & 0 & 0 & .2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Of students currently taking the class for the first time, 94% will eventually pass. 6% will eventually fail twice and be unable to earn their degree.

Of students currently taking the class for the second time time, 80% will eventually pass. 20% will eventually fail twice and be unable to earn their degree.

The solution matrix contains the same information in abbreviated form

$$\text{Solution Matrix} = \begin{array}{c} \\ C \\ R \end{array} \begin{array}{c} \begin{array}{cc} P & F \end{array} \\ \begin{bmatrix} .94 & .06 \\ .80 & .20 \end{bmatrix} \end{array}$$

Note that in this particular problem, we don't need to raise T to a "very high" power. If we find T^2, we see that it is actually equal to T^n for higher powers n. T^n becomes stable after two transitions; this makes sense in this problem because after taking the class twice, the student must have passed or is not permitted to retake it any longer. Therefore the probabilities should not change any more after two transitions; by the end of two transitions, every student has reached an absorbing state.

Absorbing Markov Chains

1. A Markov chain is an absorbing Markov chain if it has at least one absorbing state. A state i is an absorbing state if once the system reaches state i, it stays in that state; that is, $p_{ii} = 1$.
2. If a transition matrix T for an absorbing Markov chain is raised to higher powers, it reaches an absorbing state called the solution matrix and stays there. The i, j-th entry of this matrix gives the probability of absorption in state j while starting in state i.
3. Alternately, the solution matrix can be found in the following manner.
   a. Express the transition matrix in the canonical form as below.

$$T = \begin{bmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{A} & \mathbf{B} \end{bmatrix}$$

   where I_n is an identity matrix, and **0** is a matrix of all zeros.

   b. The fundamental matrix $F = (I - B)^{-1}$. The fundamental matrix helps us find the number of games played before absorption.

   c. FA is the solution matrix, whose i, j-th entry gives the probability of absorption in state j while starting in state i.
4. The sum of the entries of a row of the fundamental matrix gives us the expected number of steps before absorption for the non-absorbing state associated with that row.

SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS

1) Given the following absorbing Markov chain.

$$T = \begin{matrix} & S1 & S2 & S3 & S4 \\ S1 & 1 & 0 & 0 & 0 \\ S2 & .1 & .4 & .2 & .3 \\ S3 & 0 & 0 & 1 & 0 \\ S4 & .4 & 0 & .2 & .4 \end{matrix}$$

a) Identify the absorbing states.	b) Write the solution matrix.
c) Starting from state 4, what is the probability of eventual absorption in state 1?	d) Starting from state 2, what is the probability of eventual absorption in state 3?

2). Two tennis players, Andre and Vijay each with two dollars in their pocket, decide to bet each other $1, for every game they play. They continue playing until one of them is broke.

a) Write the transition matrix for Andre.	b) Identify the absorbing states.
c) Write the solution matrix.	d) At a given stage if Andre has $1, what is the chance that he will eventually lose it all?

3) Repeat the previous problem, if the chance of winning for Andre is .4 and for Vijay .6.

a) Write the transition matrix for Andre.	b) Write the solution matrix.
c) If Andre has $3, what is the probability that he will eventually be ruined?	d) If Vijay has $1, what is the probability that he will eventually triumph?

SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS

4) Repeat problem 2, if initially Andre has $3 and Vijay has $2.

a) Write the transition matrix.	b) Identify the absorbing states.
c) Write the solution matrix.	d) If Andre has $4, what is the probability that he will eventually be ruined?

5) The non-tenured professors at a community college are regularly evaluated. After an evaluation they are classified as good, bad, or improvable. The "improvable" are given a set of recommendations and are re-evaluated the following semester. At the next evaluation, 60% of the improvable turn out to be good, 20% bad, and 20% improvable. These percentages never change and the process continues.

a) Write the transition matrix.	b) Identify the absorbing states.
c) Write the solution matrix.	d) What is the probability that a professor who is improvable will eventually become good?

SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS

Questions 6 – 11 refer to the following:
In a professional certification program students take classes and then participate in an internships.
There are 4 states: taking classes (C), internship (I), drop out (D), and graduate (G).
If a student drops out they are never readmitted to the program.

Of those students currently taking classes, 70% have an internship the next year, 20% are still taking classes the next year, and 10% have dropped out by the next year.

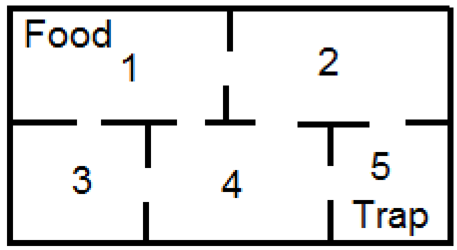
Of the students who are currently doing an internship, 65% graduate by the next year; 20% drop out by the next year, and 15% are still completing their internship the next year.

6) Write the transition matrix and indicate which are the absorbing states.	7) If a student is taking classes now: a) find the probability that the student will graduate in 2 years b) find the probability that the student will be in the internship in 2 years. c) find the probability that the student will have dropped out by 2 years from now.
8) Find the probability that a student currently doing an internship will eventually drop out.	9) Find the probability that a student taking classes now will eventually graduate.
10) If 40% of students are currently taking classes and 60% of current students are doing internships, what is the eventual long term distribution of students for graduating versus dropping out?	11) If 70% of students are currently taking classes and 30% of current students are doing internships, what is the eventual long term distribution of students for graduating versus dropping out?

SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS

12) A mouse is placed in the maze shown below, and it moves from room to room randomly. From any room, the mouse will choose a door to the next room with equal probabilities.
Once the mouse reaches room 1, it finds food and never leaves that room.
And when it reaches room 5, it is trapped and cannot leave that room.

What is the probability the mouse will end up in room 5 if it was initially placed in room 3?



13) In problem 12, what is the probability the mouse will end up in room 1 if initially placed in room 2?

SECTION 10.5 PROBLEM SET: CHAPTER REVIEW

1) Is the matrix given below a transition matrix for a Markov chain? Explain.

a) $\begin{bmatrix} .1 & .4 & .5 \\ .5 & -.3 & .8 \\ .3 & .4 & .3 \end{bmatrix}$ b) $\begin{bmatrix} .2 & .6 & .2 \\ 0 & 0 & 0 \\ .3 & .4 & .5 \end{bmatrix}$

2) A survey of computer buyers indicates that if a person buys an Apple computer, there is an 80% chance that their next purchase will be an Apple, while owners of a Windows computer will buy an Windows computer again with a probability of .70. The buying habits of these consumers are represented in the transition matrix below.

		Next Purchase	
		Apple	Windows
Present	Apple	.80	.20
Purchase	Windows	.30	.70

a) Find the probability that a present owner of an Apple computer will buy a Windows computer as his next computer.

b) Find the probability that a present owner of an Apple computer will buy a Windows computer as his third computer.

c) Find the probability that a present owner of a Windows computer will buy a Windows computer as his fourth computer.

3) Professor Trayer either teaches Finite Math or Statistics each quarter. She never teaches Finite Math two consecutive quarters, but if she teaches Statistics one quarter, then the next quarter she will teach Statistics with a 1/3 probability.

a) Write a transition matrix for this problem.

b) If Professor Trayer teaches Finite Math in the Fall quarter, what is the probability that she will teach Statistics in the Winter quarter.

c) If Professor Trayer teaches Finite Math in the Fall quarter, what is the probability that she will teach Statistics in the Spring quarter.

4) Determine whether the following matrices are regular Markov chains.

a) $\begin{bmatrix} 1 & 0 \\ .3 & .7 \end{bmatrix}$ b) $\begin{bmatrix} .2 & .4 & .4 \\ .6 & .4 & 0 \\ .3 & .2 & .5 \end{bmatrix}$

5) The transition matrix for switching academic majors each quarter by students at a university is given below, where Science, Business, and Liberal Arts majors are denoted by S, B, and A, respectively.

		TO		
		S	B	A
	S	.6	.3	.1
FROM	B	.1	.7	.2
	A	.1	.1	.8

a) Find the probability of a science major switching to a business major during their first quarter.

b) Find the probability of a business major switching to a Liberal Arts during their second quarter.

c) Find the probability of a science major switching to Liberal Artsr during their third quarter.

SECTION 10.5 PROBLEM SET: CHAPTER REVIEW

6) John Elway, the football quarterback for the Denver Broncos, called his own plays. At every play he had to decide to either pass the ball or hand it off. The transition matrix for his plays is given in the following table, where P represents a pass and H a handoff.

		Next Shot P	Next Shot H
Previous Shot	P	.6	.4
	H	.8	.2

a) If John Elway threw a pass on the initial play, what is the probability that he handed handoff on the two plays later?

b) Determine the long term play distribution.

7) Company I, Company II, and Company III compete against each other, and the transition matrix for people switching from company to company each year is given below.

		TO I	TO II	TO III
FROM	I	.6	.2	.2
	II	.3	.5	.2
	III	.3	.3	.4

a) If the initial market share is 20% for Company I, 30% for Company II and 50% for Company III, what will the market share be after the next year?

b) If this trend continues, what is the long range expectation for the market?

8) Given the following absorbing Markov chain.

$$T = \begin{array}{c} \\ S1 \\ S2 \\ S3 \\ S4 \end{array} \begin{array}{c} \begin{array}{cccc} S1 & S2 & S3 & S4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .2 & .3 & .4 & .1 \\ .4 & .1 & .1 & .4 \end{bmatrix} \end{array}$$

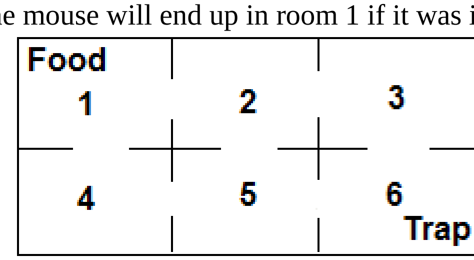
a) Identify the absorbing states.

b) Write the solution matrix.

c) Starting from state 4, what is the probability of eventual absorption in state 1?

d) Starting from state 3, what is the probability of eventual absorption in state 2?

9) A mouse placed in the maze moves from room to room randomly. From any room, the mouse will choose a door to the next room with equal probabilities. Once it reaches room 1, it finds food and never leaves that room. And when it reaches room 6, it is trapped and cannot leave that room. What is the probability that the mouse will end up in room 1 if it was initially placed in room 3?



10) What is the probability that the mouse will end up in room 6 if it was initially in room 2?

CHAPTER 11

Chapter 11: Game Theory

In this chapter, you will learn to:

1. *Solve strictly determined games.*
2. *Solve games involving mixed strategies.*

11.1 Strictly Determined Games

Game theory is one of the newest branches of mathematics. It first came to light when a brilliant mathematician named Dr. John von Neumann co-authored with Dr. Morgenstern a book titled *Theory of Games and Economic Behavior*. Since then it has played an important role in decision making in business, economics, social sciences and other fields.

In this chapter, we will study games that involve only two players. In these games, since a win for one person is a loss for the other, we refer to them as **two-person zero-sum games.** Although the games we will study here are fairly simple, they will provide us with an understanding of how games work and how they are applied in practical situations. We begin with an example.

◆***Example 1*** Robert and Carol decide to play a game using a dime and a quarter. Each chooses one of the two coins, puts it in their hand and closes their fist. At a given signal, they simultaneously open their fists. If the sum of the coins is less than 35 cents, Robert gets both coins, otherwise, Carol gets both coins.
Write the matrix for the game, determine the optimal strategies for each player, and find the expected payoff for Robert.

Solution: Suppose Robert is the row player, that is, he plays the rows, and Carol is a column player. If Robert shows a dime and Carol shows a dime, the sum will be less than thirty five cents and Robert will win ten cents. But, if Robert shows a dime and Carol shows a quarter, the sum will not be less than thirty five cents and Carol will win ten cents or Robert will lose ten cents. The following matrix depicts all four cases and their corresponding payoffs for Robert. Remember a negative value is a loss for Robert and a win for Carol.

		Carol Dime	Quarter
Robert	Dime	10	–10
	Quarter	–25	–25

The best strategy for Robert is to always show a dime because this way the worst he can do is lose ten cents. And, the best strategy for Carol is to always show a quarter because that way the worst she can do is to lose ten cents. If both Robert and Carol play their optimal strategies, Robert will lose ten cents each time. Therefore, the value of the game is negative ten cents.

In the above example, since there is only one fixed optimal strategy for each player, regardless of their opponent's strategy, we say the game possesses a **pure strategy** and is **strictly determined**.

Next, we formulate a method to find the optimal strategy for each player and the value of the game. The method involves considering the worst scenario for each player.

To consider the worst situation, the row player considers the minimum value in each row, and the column player considers the maximum value in each column. Note that the maximum value really represents a minimum value for the column player because the game matrix depicts the payoffs for the row player. We list the method below.

Finding the Optimal Strategy and the Value for Strictly Determined Games

1. Put an asterisk(*) next to the minimum entry in each row.
2. Put a box around the maximum entry in each column.
3. The entry that has both an asterisk and a box represents the value of the game and is called a **saddle point**.
4. The row that is associated with the saddle point represents the best strategy for the row player, and the column that is associated with the saddle point represents the best strategy for the column player.
5. A game matrix can have more than one saddle point, but all saddle points have the same value.
6. If no saddle point exists, the game is not strictly determined. Non-strictly determined games are the subject of the next section.

◆***Example 2*** Find the saddle points and optimal strategies for the following game.

Column Player

$$\textbf{Row Player} \begin{bmatrix} -25 & 10 \\ 10 & 50 \end{bmatrix}$$

Solution: We find the saddle point by placing an asterisk next to the minimum entry in each row, and by putting a box around the maximum entry in each column as shown below.

Column Player

$$\textbf{Row Player} \begin{bmatrix} -25^* & 10 \\ \boxed{10^*} & \boxed{50} \end{bmatrix}$$

Since the second row, first column entry, which happens to be 10, has both an asterisk and a box, it is a saddle point. This implies that the value of the game is 10, and the optimal strategy for the row player is to always play row 2, and the optimal strategy for the column player is to always play column 1. If both players play their optimal strategies, the row player will win 10 units each time.

The row player's strategy is written as $\begin{bmatrix} 0 & 1 \end{bmatrix}$ indicating that he will play row 1 with a probability of 0 and row 2 with a probability of 1.

Similarly the column player's strategy is written as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ implying that he will play column 1 with a probability of 1, and column 2 with a probability of 0.

SECTION 11.1 PROBLEM SET: STRICTLY DETERMINED GAMES

1) Determine whether the games are strictly determined. If the games are strictly determined, find the optimal strategies for each player and the value of the game.

a) $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$	b) $\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$
c) $\begin{bmatrix} -1 & -3 & 2 \\ 0 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix}$	d) $\begin{bmatrix} 2 & 0 & -4 \\ 3 & 4 & 2 \\ 0 & -2 & -3 \end{bmatrix}$
e) $\begin{bmatrix} 0 & 2 \\ -1 & -1 \\ -1 & 1 \\ 3 & 2 \end{bmatrix}$	f) $\begin{bmatrix} 5 & -3 & 2 \\ 3 & -1 & 4 \end{bmatrix}$

2) Two players play a game which involves holding out one or two fingers simultaneously. If the sum of the fingers is more than 2, Player II pays Player I the sum of the fingers; otherwise, Player I pays Player II the sum of the fingers.

a) Write a payoff matrix for Player I.	b) Find the optimal strategies for each player and the value of the game.

SECTION 11.1 PROBLEM SET: STRICTLY DETERMINED GAMES

3) A mayor of a large city is thinking of running for re-election, but does not know who his opponent is going to be. It is now time for him to take a stand for or against abortion. If he comes out against abortion rights and his opponent is for abortion, he will increase his chances of winning by 10%. But if he is against abortion and so is his opponent, he gains only 5%. On the other hand, if he is for abortion and his opponent against, he decreases his chance by 8%, and if he is for abortion and so is his opponent, he decreases his chance by 12%.

a) Write a payoff matrix for the mayor.	b) Find the optimal strategies for the mayor and his opponent.

4) A man accused of a crime is not sure whether anybody saw him do it. He needs to make a choice of pleading innocent or pleading guilty to a lesser charge. If he pleads innocent and nobody comes forth, he goes free. However, if a witness comes forth, the man will be sentenced to 10 years in prison. On the other hand, if he pleads guilty to a lesser charge and nobody comes forth, he gets a sentence of one year and if a witness comes forth, he gets a sentence of 3 years.

a) Write a payoff matrix for the accused.	b) If you were his attorney, what strategy would you advise?

11.2 Non-Strictly Determined Games

In this section, we study games that have no saddle points. This means that these games do not possess a pure strategy. We call these games **non-strictly determined games**. If the game is played only once, it will make no difference what move is made. However, if the game is played repeatedly, a **mixed strategy** consisting of alternating random moves can be worked out.

We consider the following example.

◆***Example 1*** Suppose Robert and Carol decide to play a game using a dime and a quarter. At a given signal, they simultaneously show one of the two coins. If the coins match, Robert gets both coins, but if they don't match, Carol gets both coins. Determine whether the game is strictly determined.

Solution: We write the payoff matrix for Robert as follows:

Carol

		Dime	Quarter
Robert	Dime	10	–10
	Quarter	–25	25

To determine whether the game is strictly determined, we look for a saddle point. Again, we place an asterisk next to the minimum value in each row, and a box around the maximum value in each column. We get

Carol

		Dime	Quarter
Robert	Dime	[10]	–10*
	Quarter	–25*	[25]

Since there is no entry that has both an asterisk and a box, the game does not have a saddle point, and thus it is non-strictly determined.

We wish to devise a strategy for Robert. If Robert consistently shows a dime, for example, Carol will see the pattern and will start showing a quarter, and Robert will lose. Conversely, if Carol repeatedly shows a quarter, Robert will start showing a quarter, thus resulting in Carol's loss. So a good strategy is to throw your opponent off by showing a dime some of the times and showing a quarter other times. Before we develop an optimal strategy for each player, we will consider an arbitrary strategy for each and determine the corresponding payoffs.

◆***Example 2*** Suppose in Example 1, Robert decides to show a dime with .20 probability and a quarter with .80 probability, and Carol decides to show a dime with .70 probability and a quarter with .30 probability. What is the expected payoff for Robert?

Solution: Let R denote Robert's strategy and C denote Carol's strategy.

Since Robert is a row player and Carol is a column player, their strategies are written as follows:

$$R = \begin{bmatrix} .20 & .80 \end{bmatrix} \text{ and } C = \begin{bmatrix} .70 \\ .30 \end{bmatrix}.$$

To find the expected payoff, we use the following reasoning.

Since Robert chooses to play row 1 with .20 probability and Carol chooses to play column 1 with .70 probability, the move row 1, column 1 will be chosen with (.20)(.70) = .14 probability. The fact that this move has a payoff of 10 cents for Robert, Robert's expected payoff for this move is (.14)(.10) = .014 cents. Similarly, we compute Robert's expected payoffs for the other cases. The table below lists expected payoffs for all four cases.

Move	Probability	Payoff	Expected Payoff
Row 1, Column 1	(.20)(.70) = .14	10 cents	1.4 cents
Row 1, Column 2	(.20)(.30) = .06	–10 cents	–.6 cents
Row 2, Column 1	(.80)(.70) = .56	–25 cents	–14 cents
Row 2, Column 2	(.80)(.30) = .24	25 cents	6.0 cents
Totals	1		–7.2 cents

The above table shows that if Robert plays the game with the strategy $R = \begin{bmatrix} .20 & .80 \end{bmatrix}$ and Carol plays with the strategy $C = \begin{bmatrix} .70 \\ .30 \end{bmatrix}$, Robert can expect to lose 7.2 cents for every game.

Alternatively, if we call the game matrix G, then the expected payoff for the row player can be determined by multiplying matrices R, G and C. Thus, the expected payoff P for Robert is as follows:

$$P = RGC$$

$$P = \begin{bmatrix} .20 & .80 \end{bmatrix} \begin{bmatrix} 10 & -10 \\ -25 & 25 \end{bmatrix} \begin{bmatrix} .70 \\ .30 \end{bmatrix}$$

$$P = -7.2 \text{ cents.}$$

which is the same as the one obtained from the table.

◆***Example 3*** For the following game matrix G, determine the optimal strategy for both the row player and the column player, and find the value of the game.

$$G = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

Solution: Let us suppose that the row player uses the strategy $R = \begin{bmatrix} r & 1-r \end{bmatrix}$.
Now if the column player plays column 1, the expected payoff P for the row player is

$$P(r) = 1(r) + (-3)(1 - r) = 4r - 3.$$

This can also be computed as follows:

$$P(r) = \begin{bmatrix} r & 1-r \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \text{or} \quad 4r - 3.$$

If the row player plays the strategy $\begin{bmatrix} r & 1-r \end{bmatrix}$ and the column player plays column 2, the expected payoff P for the row player is

$$P(r) = \begin{bmatrix} r & 1-r \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = -6r + 4.$$

We have two equations : $P(r) = 4r - 3$ and $P(r) = -6r + 4$

The row player is trying to improve upon his worst scenario, and that only happens when the two lines intersect. Any point other than the point of intersection will not result in optimal strategy as one of the expectations will fall short.

Solving for r algebraically, we get

$$4r - 3 = -6r + 4$$

$$r = 7/10.$$

Therefore, the optimal strategy for the row player is $\begin{bmatrix} .7 & .3 \end{bmatrix}$.

Alternatively, we can find the optimal strategy for the row player by, first, multiplying the row matrix with the game matrix as shown below.

$$\begin{bmatrix} r & 1-r \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 4r-3 & -6r+4 \end{bmatrix}$$

And then by equating the two entries in the product matrix. Again, we get r = .7, which gives us the optimal strategy $\begin{bmatrix} .7 & .3 \end{bmatrix}$.

We use the same technique to find the optimal strategy for the column player.

Suppose the column player's optimal strategy is represented by $\begin{bmatrix} c \\ 1-c \end{bmatrix}$.
We, first, multiply the game matrix by the column matrix as shown below.

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} c \\ 1-c \end{bmatrix} = \begin{bmatrix} 3c-2 \\ -7c+4 \end{bmatrix}$$

And then equate the entries in the product matrix. We get

$$3c - 2 = -7c + 4$$

$$c = .6$$

Therefore, the column player's optimal strategy is $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

To find the expected value, V, of the game, we find the product of the matrices R, G and C.

$$V = \begin{bmatrix} .7 & .3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

$$V = -.2$$

That is, if both players play their optimal strategies, the row player can expect to lose .2 units for every game.

◆***Example 4*** For the game in Example 1, determine the optimal strategy for both Robert and Carol, and find the value of the game.

Solution: Since we have already determined that the game is non-strictly determined, we proceed to determine the optimal strategy for the game. We rewrite the game matrix.

$$G = \begin{bmatrix} 10 & -10 \\ -25 & 25 \end{bmatrix}$$

Let R = $\begin{bmatrix} r & 1-r \end{bmatrix}$ be Robert's strategy, and C = $\begin{bmatrix} c \\ 1-c \end{bmatrix}$ be Carol's strategy.

To find the optimal strategy for Robert, we, first, find the product RG as below.

$$\begin{bmatrix} r & 1-r \end{bmatrix}\begin{bmatrix} 10 & -10 \\ -25 & 25 \end{bmatrix} = \begin{bmatrix} 35r-25 & -35r+25 \end{bmatrix}$$

By setting the entries equal, we get

$$35r - 25 = -35r + 25$$

or $r = 5/7$.

Therefore, the optimal strategy for Robert is $\begin{bmatrix} 5/7 & 2/7 \end{bmatrix}$.

To find the optimal strategy for Carol, we, first, find the following product.

$$\begin{bmatrix} 10 & -10 \\ -25 & 25 \end{bmatrix}\begin{bmatrix} c \\ 1-c \end{bmatrix} = \begin{bmatrix} 20c-10 \\ -50c+25 \end{bmatrix}$$

We now set the entries equal to each other, and we get,

$$20c - 10 = -50c + 25$$

or $c = 1/2$

Therefore, the optimal strategy for Carol is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

To find the expected value, V, of the game, we find the product RGC.

$$V = \begin{bmatrix} 5/7 & 2/7 \end{bmatrix}\begin{bmatrix} 10 & -10 \\ -25 & 25 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}$$

If both players play their optimal strategy, the value of the game is zero. In such case, the game is called **fair.**

SECTION 11.2 PROBLEM SET: NON-STRICTLY DETERMINED GAMES

1) Determine the optimal strategies for both the row player and the column player, and find the value of the game.

a) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$	b) $\begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$
c) $\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$	d) $\begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$

2) Find the expected payoff for the given game matrix G if the row player plays strategy R, and column player plays strategy C.

a) $G = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$ $R = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$ $C = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$	b) $G = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $R = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$ $C = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

SECTION 11.2 PROBLEM SET: NON-STRICTLY DETERMINED GAMES

3) Two players play a game which involves holding out one or two fingers simultaneously.
If the sum of the fingers is even, Player II pays Player I the sum of the fingers.
If the sum of the fingers is odd, Player I pays Player II the sum of the fingers.

a) Write a payoff matrix for Player I.	b) Find the optimal strategies for both the row player and the column player, and the value of the game.

4) In December 1995, President Clinton ordered the first of 20,000 U. S. troops to be sent into Bosnia-Herzegovina as a peace keeping force. Unfortunately, the heavy fog made visibility very poor at the Tuzla airfield, and at the same time increased the threat of sniper attacks from the Serbian forces. U. S. Air Force Col. Neal Patton, and Lt. Col. Sid Kooyman, the advance specialists, had two choices: either to send in the troops by air with the difficulties already described or by road thus exposing the troops to ambush by the Serbian forces. The Serbian army, with its limited resources, had a choice of deploying its forces near the airport or along the road route.

If the U. S. lands its troops on the airfield in the fog while the Serbs are concentrating on the road route, the payoff for U. S. is 20 points. But if the U. S. lands its troops on the airfield, and Serbians are there hiding in the fog, U. S. wins only 5 points. On the other hand, if U. S. transports its troops by road and avoids Serbs its payoff is 35 points, but if U. S. meets Serb resistance on the road route, it loses 50 points.

a) Write a payoff matrix for the game.	b) If you were Air Force Col. Neal Patton's advisor, what advice would you give him?

11.3 Reduction by Dominance

Sometimes an $m \times n$ game matrix can be reduced to a 2×2 matrix by deleting certain rows and columns. A row can be deleted if there exists another row that will produce a payoff of an equal or better value. Similarly, a column can be deleted if there is another column that will produce a payoff of an equal or better value for the column player. The row or column that produces a better payoff for its corresponding player is said to **dominate** the row or column with the lesser payoff.

◆***Example 1*** For the following game, determine the optimal strategy for both the row player and the column player, and find the value of the game.

$$G = \begin{bmatrix} -2 & 6 & 4 \\ -1 & -2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$$

Solution: We first look for a saddle point and determine that none exist. Next, we try to reduce the matrix to a 2×2 matrix by eliminating the dominated row.

Since every entry in row 3 is larger than the corresponding entry in row 2, row 3 dominates row 2. Therefore, a rational row player will never play row 2, and we eliminate row 2. We get

$$\begin{bmatrix} -2 & 6 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

Now we try to eliminate a column. Remember that the game matrix represents the payoffs for the row player and not the column player; therefore, the larger the number in the column, the smaller the payoff for the column player.

The column player will never play column 2, because it is dominated by both column 1 and column 3. Therefore, we eliminate column 2 and get the modified matrix, M, below.

$$M = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

To find the optimal strategy for both the row player and the column player, we use the method learned in section 11.2.

Let the row player's strategy be $R = \begin{bmatrix} r & 1-r \end{bmatrix}$, and the column player's be strategy be

$$C = \begin{bmatrix} c \\ 1-c \end{bmatrix}.$$

To find the optimal strategy for the row player, we, first, find the product RM as below.

$$\begin{bmatrix} r & 1-r \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -3r+1 & 6r-2 \end{bmatrix}$$

By setting the entries equal, we get

$$-3r + 1 = 6r - 2$$

or $\quad r = 1/3.$

Therefore, the optimal strategy for the row player is $\begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$, but relative to the original game matrix it is $\begin{bmatrix} 1/3 & 0 & 2/3 \end{bmatrix}$.

To find the optimal strategy for the column player we, first, find the following product.

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} c \\ 1-c \end{bmatrix} = \begin{bmatrix} -6c+4 \\ 3c-2 \end{bmatrix}$$

We set the entries in the product matrix equal to each other, and we get,

$$-6c + 4 = 3c - 2$$

or $$c = 2/3$$

Therefore, the optimal strategy for the column player is $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, but relative to the original game matrix, the strategy for the column player is $\begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$.

To find the expected value, V, of the game, we have two choices: either to find the product of matrices R, M and C, or multiply the optimal strategies relative to the original matrix to the original matrix. We choose the first, and get

$$V = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 \end{bmatrix}$$

Therefore, if both players play their optimal strategy, the value of the game is zero.

We summarize as follows:

Reduction by Dominance

1. Sometimes an $m \times n$ game matrix can be reduced to a 2×2 matrix by deleting **dominated** rows and columns.
2. A row is called a **dominated row** if there exists another row that will produce a payoff of an equal or better value. That happens when there exists a row whose every entry is larger than the corresponding entry of the dominated row.
3. A column is called a **dominated column** if there exists another column that will produce a payoff of an equal or better value. This happens when there exists a column whose every entry is smaller than the corresponding entry of the dominated row.

SECTION 11.3 PROBLEM SET: REDUCTION BY DOMINANCE

Reduce the payoff matrix by dominance.
Find the optimal strategy for each player and the value of the game.

1) $\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}$	2) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 5 & 4 \end{bmatrix}$
3) $\begin{bmatrix} 1 & 3 & -2 \\ -2 & 9 & 4 \\ -5 & 0 & 1 \end{bmatrix}$	4) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$

SECTION 11.3 PROBLEM SET: REDUCTION BY DOMINANCE

Reduce the payoff matrix by dominance.
Find the optimal strategy for each player and the value of the game.

5) $$\begin{bmatrix} 2 & 0 & -4 & 8 \\ 0 & -6 & -6 & 2 \\ 2 & -2 & 4 & 6 \\ -2 & -4 & -8 & 0 \end{bmatrix}$$

6) $$\begin{bmatrix} -1 & 3 & 2 & 4 \\ 1 & 2 & 0 & 5 \end{bmatrix}$$

7) $$\begin{bmatrix} -5 & -1 & -1 & 3 \\ -10 & 1 & 2 & -8 \\ 4 & 0 & 1 & 5 \\ 3 & -8 & 0 & 5 \end{bmatrix}$$

8) $$\begin{bmatrix} 1 & -3 & -4 & 1 \\ 1 & -4 & -1 & 3 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

SECTION 11.4 PROBLEM SET: CHAPTER REVIEW

1) Determine whether the games are strictly determined. If the games are strictly determined, find the optimal strategies for each player and the value of the game.

a) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 3 & -1 \\ 1 & 3 & -2 \\ -1 & 2 & -5 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 2 & -1 \\ 5 & 3 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 4 & 3 \\ 1 & -3 \end{bmatrix}$

2) Two players play a game which involves holding out a nickel or a dime simultaneously. If the sum of the coins is more than 10 cents, Player I gets both the coins; otherwise, Player II gets both the coins.

a) Write a payoff matrix for Player I.

b) Find the optimal strategies for each player and the value of the game.

3) Lacy's department store is thinking of having a major sale in the month of February, but does not know if its competitor store Hordstrom's is also planning one. If Lacy's has a sale and Hordstrom's does not, Lacy's sales go up by 30%, but if both stores have a sale simultaneously, Lacy's sales go up by only 5%. On the other hand, if Lacy's does not have a sale and Hordstrom's does, Lacy's loses 5% of its sales to Hordstrom's, and if neither of the stores has a sale, Lacy's experiences no gain in sales.

a) Write a payoff matrix for Lacy's.

b) Find the optimal strategies for both stores.

4) Mr. Halsey has a choice of three investments: Investment A, Investment B, and Investment C. If the economy booms, then Investment A yields 14% return, Investment B returns 8%, and Investment C 11%. If the economy grows moderately, then Investment A yields 12% return, Investment B returns 11%, and Investment C 11%. If the economy experiences a recession, then Investment A yields a 6% return, Investment B returns 9%, and Investment C 10%.

a) Write a payoff matrix for Mr. Halsey.

b) What would you advise him?

5) Mr. Thaggert is trying to decide whether to invest in stocks or in CD's(Certificate of deposit). If he invests in stocks and the interest rates go up, his stock investments go down by 2%, but he gains 1% in his CD's. On the other hand if the interest rates go down, he gains 3% in his stock investments, but he loses 1% in his CD's.

a) Write a payoff matrix for Mr. Thaggert.

b) If you were his investment advisor, what strategy would you advise?

SECTION 11.4 PROBLEM SET: CHAPTER REVIEW

6) Determine the optimal strategies for both the row player and the column player, and find the value of the game.

a) $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ b) $\begin{bmatrix} -2 & 2 \\ 5 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 5 \\ 4 & -1 \end{bmatrix}$ d) $\begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix}$

7) Find the expected payoff for the given game matrix G if the row player plays strategy R, and the column player plays strategy C.

a) $G = \begin{bmatrix} 3 & 5 \\ 4 & -1 \end{bmatrix}$ $R = [\,1/2 \quad 1/2\,]$ $C = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$

b) $G = \begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix}$ $R = [\,2/3 \quad 1/3\,]$ $C = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

8) A group of thieves are planning to burglarize either Warehouse A or Warehouse B. The owner of the warehouses has the manpower to secure only one of them. If Warehouse A is burglarized the owner will lose $20,000, and if Warehouse B is burglarized the owner will lose $30,000. There is a 40% chance that the thieves will burglarize Warehouse A and 60% chance they will burglarize Warehouse B. There is a 30% chance that the owner will secure Warehouse A and 70% chance he will secure Warehouse B. What is the owner's expected loss?

9) Two players play a game which involves holding out a nickel or a dime. If the sum of the coins is odd, Player I gets both the coins, and if the sum of the coins is even, Player II gets both the coins. Determine the optimal strategies for both the row player and the column player, and find the expected payoff.

10) A football quarterback has to choose between a pass play or a run play depending on how the defending team is going to react. If he chooses a pass play and the defending team is expecting a pass, he expects to gain 4 yards, but if the defending team is expecting a run, he gains 20 yards. On the other hand, if he calls a run play and the defending team expects a pass, he gains 7 yards, and if he calls a run play and the defending team expects a run, he loses 2 yards. If you were the quarterback, what would your strategy be?

11) The Watermans go fishing every weekend either at Eel River or at Snake River. Unfortunately, so do the Nelsons. If both families show up at Eel River, the Watermans can hope to catch only 3 fish, but if the Watermans fish at Eel River and the Nelsons at Snake River, the Watermans can catch as many as 12 fish. On the other hand, if both families fish at Snake river, the Watermans can catch about 5 fish, and if Watermans fish at Snake river while the Nelsons fish at Eel river, the Watermans can catch up to 15 fish. Determine a mixed strategy for the Watermans, and the expected payoff.

SECTION 11.4 PROBLEM SET: CHAPTER REVIEW

12) Terry knows there is a quiz tomorrow, but does not remember whether it is in his math class or in his biology class. He has time to study for only one subject. If he studies math and there is a quiz in it, he gains 10 points and even if there is no quiz he gains two points for acquiring the extra knowledge which he will apply towards the final exam. If he studies biology and there is a quiz in it, he gains ten points but there is no gain if there is no quiz. Determine a mixed strategy for Terry, and the expected payoff.

13) Reduce the payoff matrix by dominance. Find the optimal strategy for each player and the value of the game.

a) $\begin{bmatrix} -3 & 1 & 2 \\ -3 & 5 & 3 \\ 2 & 4 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 4 \\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 3 & 9 & 7 \\ -7 & -5 & -3 & 5 \\ -1 & 4 & 5 & 8 \\ -3 & -5 & 1 & -1 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 3 & 1 & 5 \\ -2 & 2 & 1 & 3 \end{bmatrix}$

e) $\begin{bmatrix} 0 & 3 & 2 & 1 \\ 0 & 2 & 1 & -7 \\ -4 & -9 & 5 & 4 \\ 4 & -7 & 6 & 6 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & -3 & 0 & 4 \\ 2 & -2 & -3 & 2 \end{bmatrix}$

Chapter 12: Calculator Instructions: TI-83+ and TI-84+

Section 12.1: Calculator: Graphing Equations:

Check that calculator is in "function" mode.

Press the MODE key.

A screen with showing settings will appear.
The fourth line on that screen shows **FUNC PAR POL SEQ**

FUNC should be the only item highlighted. **FUNC PAR POL SEQ**
If any other item is highlighted, use the left arrow key to move the cursor to select **FUNC** and press ENTER.

Graphing a function when you know its equation:

Press the Y= key to access the equation editor

Enter the equation into the Y= equation editor as equation Y1

To insert the variable X into the equation, use the X, T, θ,n key

Press GRAPH if you want to graph in the current graphing window stored in your calculator

Determining a Graphing Window

Here are some options to consider for determining the graphing window. The standard graphing window may work well for some graphs, but often will not work well for application problems. For application problems, the situation in the problem may provide clues to selecting an appropriate window; some guessing and trial and error may be involved.

Standard Graphing Window

Press ZOOM arrow down to **6: Standard** and press ENTER
This sets the window to the standard graphing window with $-10 \le x \le 10$ and $-10 \le y \le 10$.
If you press WINDOW, after you have pressed ZOOM 6: Standard, it will show Xmin = –10, Xmax = 10, Ymin = –10 and Ymax = 10.

Manually Selecting a Graphing Window:

Read the problem to see if the wording gives clues about an appropriate window such as giving values for points that are on the line or curve or stating some x or y values of interest in the problem.

To manually set a window, press WINDOW, input the desired viewing domain and range using Xmin, Xmax, Ymin and Ymax. Then press GRAPH

After looking at the graph, adjust the window as needed if it does not show an appropriate view.

Example: It costs \$3500 to produce 100 items and it costs \$5500 to produce 200 items. Find and graph the cost function.

Determining a window:
In this example x = the number of items and y = the cost.
The given information indicates that the x values should go up to or beyond 200 items and the y values should go up to or beyond \$5500. In this context, only non-negative values of x and y are needed; the number of items produced and the production cost don't make sense as negative numbers.

We might decide to select the graphing window as $0 \le x \le 250$, $0 \le y \le 6000$.
Use Xmin =0, Xmax=250, Ymin=0 Ymax =6000.
Use Xscl and Yscl to specify the desired interval for the tick marks shown on each axis.

Press WINDOW and input the appropriate values for Xmin, Xmax, Ymin, Ymax.

Then press GRAPH

Xmin=0	minimum x value for the window
Xmax=250	minimum x value for the window
Xscl=50	select interval for tick marks shown on the x axis
Ymin=0	minimum y value for the window
Ymax=6000	minimum y value for the window
Yscl=1000	interval for tick marks shown on the y axis
Xres=1	keep this at 1; it affects the quality of the drawn graph

Example: Vijay invests \$12000 at 5% interest for 10 years, with annual compounding.

Determining a window:
The problem refers to a time period of 10 years, so Xmax should be at least 10.
We can select a higher value for x if we want we want the value x = 10 years to be more centered in the window.

We might decide to use $0 \le x \le 15$, letting Xmax = 15.

Ymax should be at least as large as the amount Vijay will have at the end of 10 years, but in this case should be higher because we decided to include x values up to 15.
We can calculate $12000(1.05)^{10} = 19{,}547$ and $12000(1.05)^{15} = 24{,}947$.
It appears that an appropriate window would have $0 \le y \le 25000$, so Ymax = 25000.

Press WINDOW and input the appropriate values for Xmin, Xmax, Ymin, Ymax.

Then press GRAPH

Xmin=0	minimum x value for the window
Xmax=15	minimum x value for the window
Xscl=1	select interval for tick marks shown on the x axis
Ymin=0	minimum y value for the window
Ymax=25000	minimum y value for the window
Yscl=5000	interval for tick marks shown on the y axis
Xres=1	keep this as 1

Adjusting the Graphing Window:

Once we've guessed at an initial window, we can adjust the window as needed by trial and error. Press WINDOW and enter new values for Xmin, Xmax, Ymin, Ymax to find a better view. It usually takes a few tries, but it gets easier with more experience.

Use the ZOOM menu to Zoom In or Zoom Out to adjust the window. Y Zoom In or Zoom Out repeatedly, if needed. Zoom adjusts the x and y boundaries in the window proportionally, zooming in or out by a "zoom factor" of 4 (default value). The "zoom factor" can be changed by pressing ZOOM, selecting the **MEMORY** menu and selecting **4: SetFactors**

Press ZOOM: use **2:Zoom In** or **3:Zoom Out** to adjust the view.
After pressing **2:Zoom In** or **3:Zoom Out** , the press ENTER.

The calculator will show the graph in the current graphing window with the cursor located at the center .

- To zoom in or zoom out with the graphing window centered at the same location, press ENTER.
- To move the center to a different location when zooming in or out, use the arrow keys to move the cursor to the point you want at the center of your graph. Press ENTER when the cursor is located at the desired center of the window, and your calculator will zoom as instructed.

Finding an intersection of two lines or curves

Press Y= and enter the functions into Y1 and Y2, using the X,T,θ,n key to enter the variable X

Graph the functions in a window that shows the point where they intersect.

If the intersection of the two functions does not show in the window, adjust the graphing window to see the intersection point. Adjust the window manually or Zoom Out, as per instructions above.

If the intersection of the two functions is visible in the window, continue below:

Press 2nd CALC

Select **5: Intersect** from the menu and press enter

The screen shows both functions. A point on the graph of equation Y1 is highlighted showing the prompt **First curve?** Press ENTER to select that function.

Next, a point on the graph of equation Y2 is highlighted showing the prompt **Second curve?** Press ENTER to select that function.

The prompt **Guess?** will appear. Use the right and left arrow keys to move the cursor along the line until it is near the point of intersection; press ENTER.

The screen will show the estimated coordinates of the intersection point.

Section 12.2: Calculator: Matrices

Entering a matrix into the calculator:

Press [2nd] [MATRIX]. The screen will display the Matrix menu. Use the right arrow key twice to select the **EDIT** menu. From the **EDIT** menu, use the down arrow to move the cursor to select the matrix name desired from the menu, and press [ENTER].

The matrix input screen will appear.

Dimensions:
Enter the dimensions of the matrix size as rows × columns.
The cursor is positioned to the left of the ×. Type the number of rows to the left of the ×, press [ENTER] and the cursor will move to the right of the ×. Type the number of columns to the right of the × and press [ENTER] again. The shape of the matrix adjusts on the screen to show the requested number of rows and columns.

Check that the shape matches the desired matrix; if it does not, then return to the top row and adjust the dimensions. If the matrix is too large to fit the screen, use the arrow keys to scroll right or down to see the remaining rows and columns.

Matrix Entries:
Input the matrix entries, press [ENTER] after each.
The cursor scrolls through the matrix by moving across each row from left to right and then down to the next row. Using the arrow keys to move the cursor instead of pressing [ENTER] may result in a value not being stored in the calculator memory.

After entering all data, proofread the matrix. Use the arrow keys to move the cursor to the proper position if any entries need correction. Press [ENTER] after each correction.

Example: Suppose we want to input the matrix $C = \begin{bmatrix} -4 & 0 & 3 \\ 1 & 2 & -5 \end{bmatrix}$

After following the instructions above, the matrix edit screen would show

MATRIX [C] 2 x 3

$$\begin{bmatrix} -4 & 0 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Using Matrix Names

Matrices are indicated by a capital letter between square brackets, such as [A] or [C].

This is a single symbol and must be selected from the Matrix menu any time a matrix is needed.

If we type three characters **[** followed by **A** followed by **]** the calculator will not interpret the three symbol sequence [A] as a matrix and will not call up the matrix.
We MUST use the Matrix menu that shows the matrix names to call up a matrix.

Calling a Matrix to the home screen or to use in a calculation

To call a matrix back to the screen to view or use in a calculation, at any time after editing, use the MATRIX menu to enter the matrix name.

Press 2^{nd} MATRIX. Use the down arrow to select the name of the matrix and press ENTER.

This selects a matrix name to use only. It does not permit editing the matrix. To edit the matrix, arrow right over to the EDIT menu as described earlier.

Example: To view matrix C that we entered earlier, the procedure is:

2^{nd} MATRIX arrow down to 3: [C] and press ENTER ; the Matrix Menu screen shows:

```
NAMES MATH EDIT
1: [A]
2: [B]
3: [C]
4: [D]
5: [E]
```

Arrow down to use the curser to select 3: [C] , press ENTER; the home screen shows

```
[C]
```

Press ENTER again; the screen shows

[C]

$$\begin{bmatrix} -4 & 0 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Matrix Transpose

Transposing a matrix means switching the rows and columns. Columns become rows and rows become columns. If an r x c matrix had r rows and c columns, its transpose will dimension c x r with c rows and r columns

Enter the matrix into the calculator using the MATRIX EDIT screen.

Press 2^{nd} MATRIX and use down arrow key to select matrix name from the **NAMES** menu.

Press 2^{nd} MATRIX and use down arrow key to select t the **MATH** menu; use down arrow key to select **2:T**. Press ENTER
The screen shows the command such as $[A]^T$. Press ENTER

The calculator will display the transpose obtained by interchanging rows and columns

To store this matrix for later use, press the STO> key and access the matrix names list to select the name of the matrix to store it to.

Error Messages

ERR: DIM MISMATCH or ERR: INVALID DIM indicate a matrix operation that is not permissible due to the dimensions of the matrix or matrices. ERR: SINGULAR MATRIX indicates a square matrix that does not have an inverse.

Matrix Arithmetic

Use arithmetic keys + , –, × for matrix addition and subtraction, scalar multiplication and matrix multiplication. Use the matrix names menu to insert the matrix names. Press [ENTER] after you have completed the command for your calculation. To store answer for later use, press the [STO>] key and use matrix names menu to select the name of the matrix to store it to.

Example: Scalar Multiplication :

3*[2nd] [MATRIX] **1:[A]** [STO >] [2nd] [MATRIX] **3: [C]** [ENTER]
Screen shows: 3[A] →[C] and shows the matrix that results from this operation.
What the calculator does: Multiplies each entry in matrix A by 3. Stores result in Matrix C

Example: Matrix Addition

[2nd] [MATRIX] **1:[A]** + [2nd] [MATRIX] **2:[B]** [STO>] [2nd] [MATRIX] **3: [C]** [ENTER]
Screen shows: [A] +[B] →[C] and shows the matrix that results from this operation.
What the calculator does: Adds entries in matrices A and B. Stores the result in matrix C

Matrix Subtraction: see matrix addition above; use subtraction operation – instead of +

Example: Several operations in the same step: 4A–3B

4* [2nd] [MATRIX] **1:[A]** – 3* [2nd] [MATRIX] **2:[B]** [STO>] [2nd] [MATRIX] **3: [C]** [ENTER]
Screen shows: 4*[A] – 3*[B] →[C]
What the calculator does: Performs scalar multiplication and subtraction to find matrix 4A – 3B . Stores the result in matrix C

Example: Several operations in the same step: 4(A+B)

4*([2nd] [MATRIX] **1:[A]** +[2nd] [MATRIX] **2:[B]**) [STO>] [2nd] [MATRIX] **3: [C]** [ENTER]
Screen shows: 4*([A]+[B]) →[C]
What the calculator does: Adds matrices A and B to find matrix A+B then performs scalar multiplication using the matrix sum to find 4(A+B). Stores the result in matrix C

Example: Matrix Multiplication: AB

[2nd] [MATRIX] **1:[A** *[2nd] [MATRIX] **2:[B]** [STO >] [2nd] [MATRIX] **3: [C]** [ENTER]
Screen shows: [A]*[B] →[C]
What the calculator does: Uses matrix multiplication to multiply matrix A by matrix B to find the matrix product AB. Stores the result in matrix C

Example: Raise a Matrix to a power: A^4

[2nd] [MATRIX] **1:[A** ^4] [STO >] [2nd] [MATRIX]**3: [C]** [ENTER]
Screen shows: [A] ^4→[C]
What the calculator does: Calculates A^4 as A*A*A*A . Stores the result in matrix C

Matrix Inverse

Enter the matrix into the calculator using the MATRIX EDIT screen.

Press [2nd] [MATRIX] and use down arrow key to select matrix name from the **NAMES** menu.

Press the [x^{-1}] key (for inverse). Press [ENTER]

The screen shows the command such as **$[A]^{-1}$**. Press [ENTER] Calculator will display the inverse.

To store this matrix for later use, press the [STO>] key and access the matrix names list to select the name of the matrix to store it to

Row Operations: calculator commands

The commands below explain the row operations that are built into the calculator.
They are on the 2nd MATRIX **MATH** menu.

Be careful to enter the information into the command exactly as explained below.
If information is put in the wrong order, the calculator will do exactly what it is told to do and give an incorrect answer for what was intended; the calculator follows commands but cannot read read minds to know what was intended if the information is not entered correctly.

To store the resulting matrix for later use, press the STO> key and access the matrix names menu to select the name of the matrix to store the result.

rowSwap(*matrixname, row r, row s*) interchanges (swaps) row r and row s

Example: rowSwap (*[A]*,*2*,*3*)
uses matrix A and interchanges row 2 and row 3.

row+(*matrixname, row r, row s*) adds each entry in row r to the corresponding entry in row s and stores the result in row s

Example: row+(*[A] 3*,*2*)
adds row 3 to row 2 and stores the result in row 2.

***row(*constant, matrixname, row r*)** multiplies all entries in row r by a constant number

Example: *row (*5*, *[A]*)
multiplies every entry in matrix A by the number 5.

Example: *row (*–8*, *[A]*)
multiplies every entry in matrix A by the number –8.

***row+(*constant, matrixname, row r, row s*)** multiplies row r by a constant and adds the resulting row to row s, storing the result in row s.

Example *row+(*–4*, *[A]*,*2*,*3*)
uses matrix A; it multiplies row 2 by the constant *(*–4) and adds this result to row 3, storing the result in row 3.

Reduced Row Echelon Form

This finds Reduced Row Echelon form in one easy fast step. However in many cases your instructor may require you to show step by step work, so be sure you use this only if the work is not required to be shown. But, even if you need to do the work to show step by step work, this command is helpful for checking your answer.

Enter the matrix into the calculator using the MATRIX EDIT screen.

Press 2nd QUIT.
Press 2nd MATRIX ; use the right arrow key to select the **MATH** menu, use down arrow to move the cursor to select **rref(** . Press ENTER.
Press 2nd MATRIX ; use down arrow to select matrix name from **NAMES** menu. Press ENTER

Close the parentheses) . Press ENTER

The home screen will show a command such as: **rref ([A])**.
Press ENTER

The screen will display reduced row echelon form of this matrix.

Section 12.3: Finance App: TVM Solver Instructions

The Finance App has many functions built into it. We examine only the TVM (Time Value of Money) solver here. The TVM Solver is useful for most problems in this textbook.

To learn to use the other functions in the Finance app, search on the internet for instructions for the other functions or consult the Texas Instruments calculator manual. Each function has its own syntax for entering required information. Any errors in the syntax will result in an error in the answer. If using these functions, be careful to use correct syntax and order of the inputs.

Accessing the Time Value of Money calculator in the Finance APP:

Press [2nd] [App]

Use down arrow to select **Finance** (usually at or near the top of the list of apps)

Select **1:TVM Solver** from the menu; press [ENTER]

In the TVM solver enter all the known values. If the value is not known or does not apply to the problem, enter 0 (exception to entering 0 is P/Y and C/Y – see below).

Use of Signs to Indicate Direction of Flow of Money:

The calculator uses sign to denote the direction of flow of money.
Visualize the calculation as a transaction occurring between two entities A and B.

- If A lends B $1000, then the present value PV of $1000 is positive from the point of view of B, as money coming in; then the payments PMT that B makes to A to repay the loan will appear as negative, indicating money B pays out.
- However if you view the transaction from the point of view of A, then $1000 would be negative as money paid out by A, and the periodic payments made by B to repay the loan would be positive as they represent money flowing in to A.

It is not important which viewpoint you adopt, but it is important to be internally consistent about the signs showing the correct direction for the flow of money in the transaction.

Variables used by the TVM solver screen

N = the total number of compounding periods (N = nt using the notation of this textbook)

I% = enter the interest rate expressed as a percent

PV = present value

PMT = periodic payment

FV = future value (accumulated value)

P/Y = number of payments per year

C/Y = number of compounding periods per year

PMT: END BEGIN – highlight to indicate whether payments are made at the end of beginning of a payment period.

Finding the solution after entering all input

Use arrow keys to move the cursor to highlight the variable you want to calculate.
Press [ALPHA] [ENTER] to Solve.
The calculator will calculate the value of the variable that is highlighted

Section 12.4: Finance App: TVM Solver Examples

Example: To calculate the future (accumulated) value of an investment of $3000 invested at 4.2% interest compounded quarterly for 5 years.

N = 20 (there are 4x5 = 20 compounding periods total in this example)

I% = 4.2

PV = 3000

PMT = 0 (the problem does not have any periodic payments)

FV = input 0 for now; we will solve for this soon.

P/Y = 4

C/Y = 4

PMT: [END] BEGIN

Use arrow keys to move cursor to highlight FV;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for FV.

Example: To calculate the present value needed now to accumulate to $8000 at the end of 5 years if invested at 4.2% interest compounded quarterly.

N = 20 (there are 4x5 = 20 compounding periods total in this example)

I% = 4.2

PV = input 0 for now; we will solve for this soon

PMT = 0 (the problem does not have any periodic payments)

FV = 8000

P/Y = 4

C/Y = 4

PMT: [END] BEGIN

Use arrow keys to move cursor to highlight PV;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for PV.

Example: To calculate the present value now of periodic payments of $500 monthly for 10 years invested at 6% interest compounded monthly, assuming payments are made at the end of each period.

N = 120 (there are 12x10 = 120 compounding periods total in this example)

I% = 6

PV = input 0 for now; we will solve for this soon.

PMT = 500

FV = 0

P/Y = 12

C/Y = 12

PMT: [END] BEGIN

Use arrow keys to move cursor to highlight PV;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for PV.

Finance App: TVM Solver Examples, continued

Example: Find the accumulated (future) value at the end of 10 years of periodic payments of $500 monthly for 10 years invested at 6% interest compounded monthly, assuming payments are made at the end of each period.

N = 120 (there are 12x10 = 120 compounding periods total in this example)
I% = 6
PV = 0
PMT = 500
FV = input 0 for now; we will solve for this soon.
P/Y = 12
C/Y = 12
PMT: [END] BEGIN

Use arrow keys to move cursor to highlight FV;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for FV.

Example: To calculate the amount of a quarterly payment needed into a sinking fund for 4 years in order to accumulate to a future value of $50000 at the end of 4 years, if invested at 7.5% compounded quarterly, assuming payments are made at the end of each quarter.

N = 16 (there are 4x4 = 16 compounding periods total in this example)
I% = 7.5
PV = 0
PMT = input 0 for now; we will solve for this soon
FV = 50000
P/Y = 4
C/Y = 4
PMT: [END] BEGIN

Use arrow keys to move cursor to highlight PMT;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for PMT.

Finance App: TVM Solver Examples, continued

Example: To calculate the amount of a monthly payment needed at the end of each month for 15 years in order to repay a mortgage loan of $150000, if the loan interest rate is 5.3% compounded monthly.

N = 180 (there are 12x15 = 180 compounding periods total in this example)

I% = 5.3

PV = 150000

PMT = input 0 for now; we will solve for this soon

FV = 0

P/Y = 12

C/Y = 12

PMT: [END] BEGIN

Use arrow keys to move cursor to highlight PMT;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for PMT.

Example: To calculate the outstanding balance at the end of 10 years for a 30 year mortgage with monthly payments of $2300, if the loan interest rate is 4.7% compounded monthly.

Note that the mortgage has a loan period of 30 years. We are asked to find the outstanding balance at the end of 10 years. The mortgage still has 30–10=20 years of payments remaining; t = 20 for this situation. The outstanding balance is the present value of the remaining 20 years of payments.

N = 240 (there are 12x20 = 240 compounding periods total in this example)

I% = 4.7

PV = input 0 for now ; we will solve for this soon.

PMT = 2300 (the problem does not have any periodic payments)

FV =0

P/Y = 12

C/Y = 12

PMT: [END] BEGIN

Use arrow keys to move cursor to highlight PV;
Press [ALPHA] [ENTER] to solve.
The calculator will then show the value for PV.

Section 12.5: Calculator: Factorials, Combinations, Permutations

Factorials, Combinations, and Permutations are found on the MATH PRB menu

Press [MATH]. Use right arrow key to move cursor to select PRB (probability) menu at the top of the screen. This menu contains the following items useful for combinatorics and probability:

2:nPr permutations
3:nCr combinations
4:! factorial

Example: Find 6!

6 [MATH] PRB 4:!
Screen shows 6!
Press [ENTER]. The answer is 720

Example: Find 8!/3!

(8 [MATH] PRB 4:!)* (3 [MATH] PRB 4:!)
Screen shows 8!/3!
Press [ENTER]. The answer is 6720

Example: Find 8P5

8 [MATH] PRB 2:nPr 5
Screen shows 8P5
Press [ENTER]. The answer is 6720

Example: Find 8C5

8 [MATH] PRB 3:nCr 5
Screen shows 8C5
Press [ENTER]. The answer is 56

Example: Find 8C5 using factorials and the definition of 8C5 instead of the 8C5 key.
Note that the parentheses in the denominator are crucial for correct evaluation.

(8 [MATH] PRB 4:!) / (3 [MATH] PRB 4:!* 5 [MATH] PRB 4:!)
Screen shows 8!/(3!*5!)
Press [ENTER]. The answer is 56

Example: Find $\frac{9!}{2!3!4!}$. In the calculator, do $\frac{9!}{(2!*3!*4!)}$.

Note that the parentheses in the denominator are crucial for correct evaluation.
This example is the number of different (unique) arrangements of the string of letters AABBBCCCC.

(9 [MATH] PRB 4:!) / (2 [MATH] PRB 4:! * 3 [MATH] PRB 4:! * 4 [MATH] PRB 4:!)
Screen shows 9!/(2! *3! *4!)
Press [ENTER]. The answer is 1260